Computing H_∞ Norms of Time-Delay Systems

Suat Gumussoy and Wim Michiels
Department of Computer Science, K.U.Leuven

H_∞ Norm of a System

H_∞ Norm of a stable system G is defined as:

$$\|G\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_1(G(j\omega))$$

- H_∞ Norm is a robustness measure of the system
- Therefore, H_∞ Norm computation and H_∞ Norm optimization are widely used in Robust Control

Problem Definition

Compute H_∞ Norm of a stable time-delay system G:

$$G(s) = C\left(sI - A_0 - \sum_{i=1}^{m} A_i e^{-\tau_i s}\right)^{-1} B + D$$

- Delays are positive, matrices with appropriate dimensions

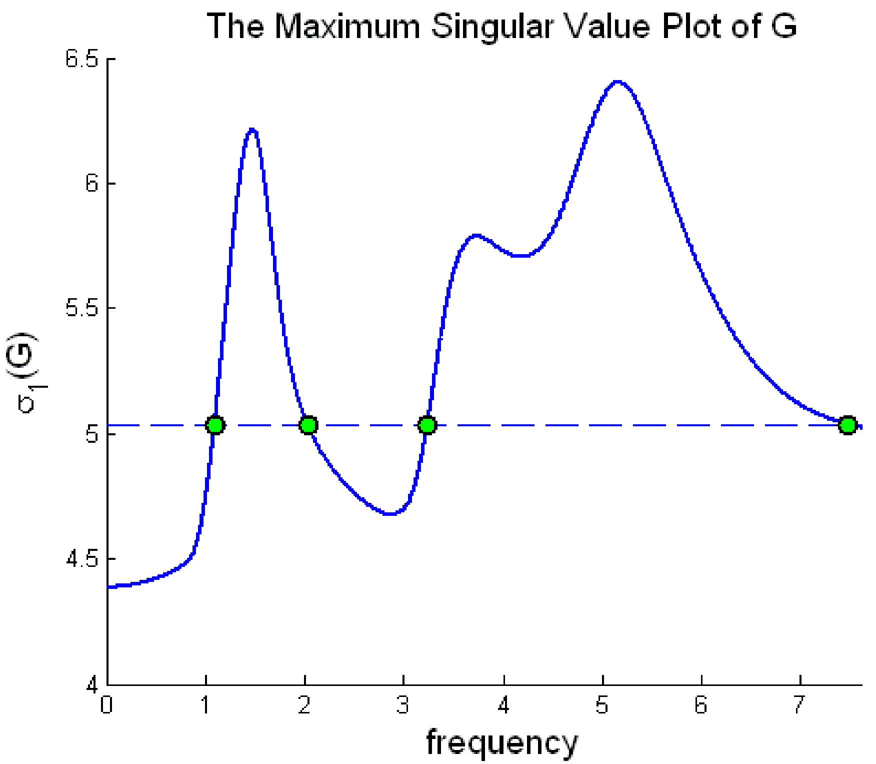
H_∞ Norm computation: Finite Dimensional Case

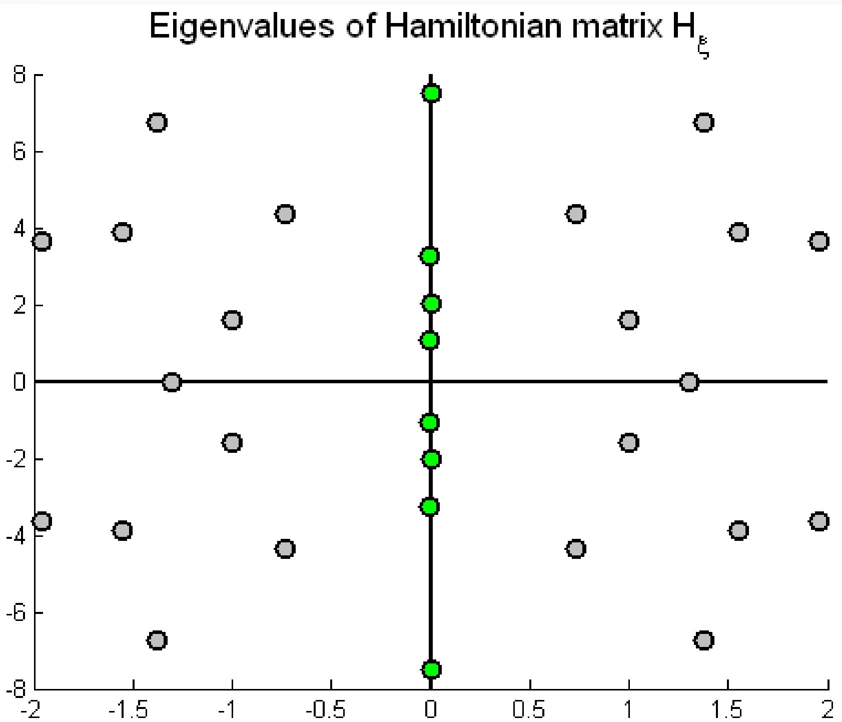
[Byers'88] Singular values of G and eigenvalues of the Hamiltonian matrix H of G have the relation:

$$\sigma_i(G(j\omega_0)) = \xi \iff \det(j\omega_0 I - H_\xi) = 0$$

Example:

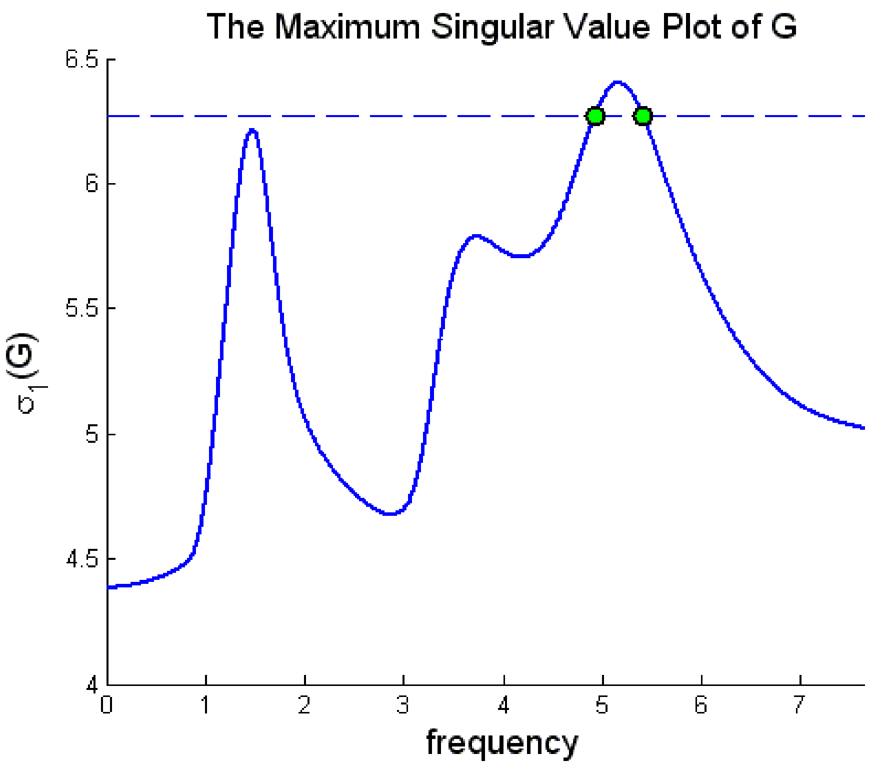
$$\sigma_i(G(j\omega_0) = \xi \iff \det(j\omega_0 I - H_\xi) = 0$$

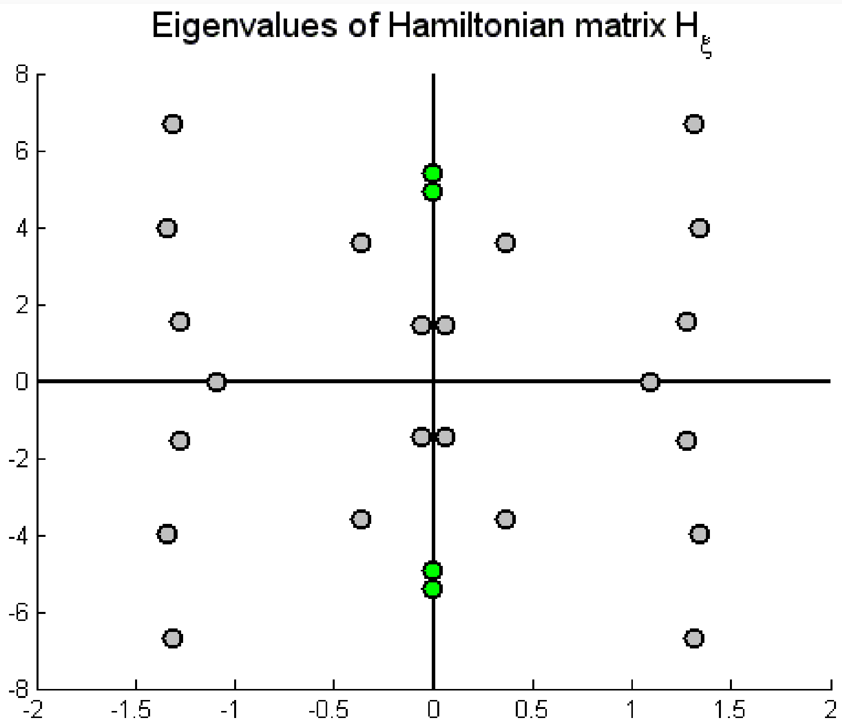




Example:

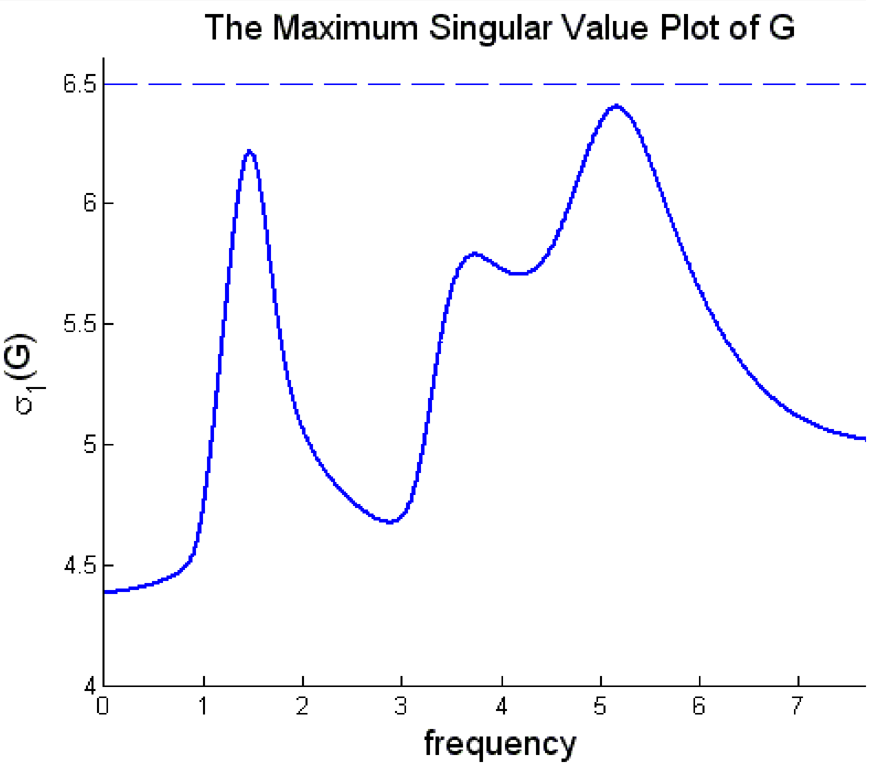
$$\sigma_i(G(j\omega_0)) = \xi \iff \det(j\omega_0 I - H_\xi) = 0$$

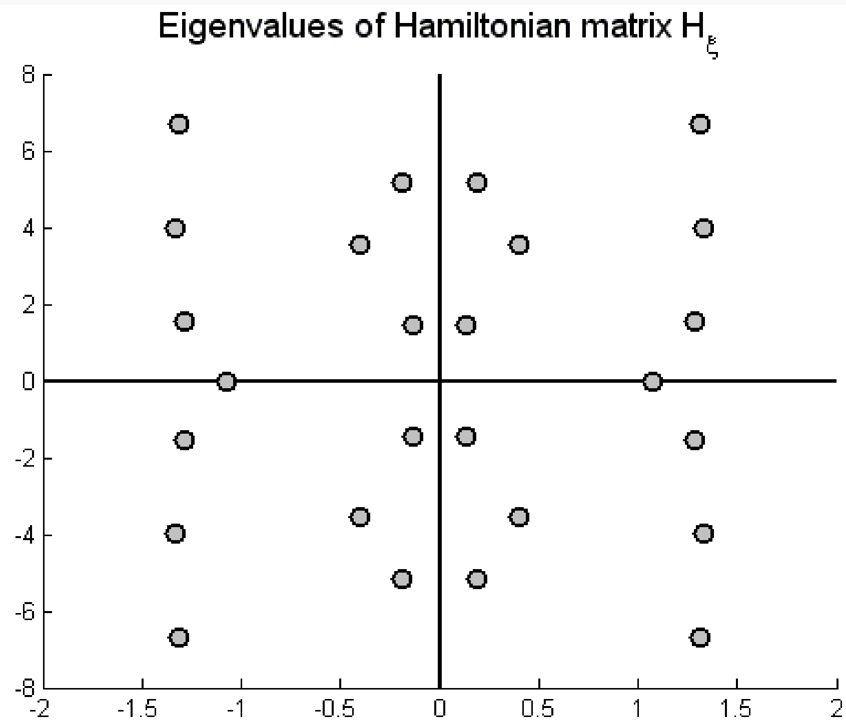




Example:

$$\sigma_i(G(j\omega_0)) = \xi \iff \det(j\omega_0 I - H_\xi) = 0$$





The Connection for Time-Delay Systems

[Thm 2.1] Let ζ> 0 be such that the matrix $D_\xi := D^T D - \xi^2 I$ is non-singular. Singular values of G and eigenvalues of the Hamiltonian-like operator L_ζ of G have the relation:

$$\sigma_i(G(j\omega_0)) = \xi \iff L_\xi u = j\omega_0 \, u$$

where L$_\zeta$ on $X := \mathcal{C}([-\tau_{\max}, \ \tau_{\max}], \mathbb{C}^{2n})$ is defined by

$$\mathcal{D}(\mathcal{L}_\xi) = \{\phi \in X : \ \phi' \in X,$$

$$\phi'(0) = M_0\phi(0) + \sum_{i=1}^{m}(M_i\phi(-\tau_i) + M_{-i}\phi(\tau_i))\},$$

$$\mathcal{L}_\xi\phi = \phi', \ \phi \in \mathcal{D}(\mathcal{L}_\xi)$$

with

$$M_0 = \begin{bmatrix} A_0 - BD_\xi^{-1}D^TC & -BD_\xi^{-1}B^T \\ \xi^2C^TD_\xi^{-T}C & -A_0^T + C^TDD_\xi^{-1}B^T \end{bmatrix},$$

$$M_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{-i} = \begin{bmatrix} 0 & 0 \\ 0 & -A_i^T \end{bmatrix}, \quad 1 \le i \le N.$$

[Corollary]

$$\|G\|_{\mathcal{H}_\infty} = \sup\{\xi \in \mathbb{R}_+ : \text{ operator } \mathcal{L}_\xi \text{ has an eigenvalue on the imaginary axis}\}$$

Properties of L_ζ

- infinite dimensional linear operator
- has infinitely many eigenvalues, finite on imaginary axis
- eigenvalues are symmetric with respect to imaginary axis
- eigenvalues of the discretized linear operator can be used as an approximate result

$$\|G\|_{\mathcal{H}_\infty} \approx \sup\{\xi \in \mathbb{R}_+ : \text{ matrix } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$$

Main Idea in a nutshell

Prediction step

Calculate the approximate H_∞ norm as

$$\|G\|_{\mathcal{H}_\infty} \approx \sup\{\xi \in \mathbb{R}_+ : \text{ matrix } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$$

Correction step

correct the approximate results from the prediction step

Discretizing the Linear Operator L_ζ

Replace the continuous space X with the space X_N of discrete functions

$$X := \mathcal{C}([-\tau_{\max}, \ \tau_{\max}], \mathbb{C}^{2n})$$

$$x = \begin{bmatrix} x_{-N} \\ \vdots \\ x_0 \\ \vdots \\ x_N \end{bmatrix}$$

$$-\tau_{\max} \leq \theta_{N,-N} < \ldots < \theta_{N,0} = 0 < \cdots < \theta_{N,N} \leq \tau_{\max}$$

$$x_i = \phi(\theta_{N,i}) \in \mathbb{C}^{2n}, \quad i = -N, \ldots, N$$

Let $P_N x$, x in X_N be the unique C^{2n} valued interpolating polynomial of degree less than or equal to 2N satisfying

$$\mathcal{P}_N x(\theta_{N,i}) = x_i, \quad i = -N, \ldots, N$$

The operator L_ξ over X can be approximated with the matrix

$L_\xi^N: X_N \longrightarrow X_N$

$$\left(\mathcal{L}_\xi^N\ x\right)_i = \left(\mathcal{P}_N x\right)'(\theta_{N,i}), \quad i = -N, \ldots, -1,$$

$$\left(\mathcal{L}_\xi^N\ x\right)_0 = M_0 \mathcal{P}_N x(0) + \sum_{i=1}^m \left(M_i \mathcal{P}_N x(-\tau_i) + M_{-i} \mathcal{P}_N x(\tau_i)\right)$$

$$\left(\mathcal{L}_\xi^N\ x\right)_i = \left(\mathcal{P}_N x\right)'(\theta_{N,i}), \quad i = 1, \ldots, N.$$

$$\mathcal{D}(\mathcal{L}_\xi) = \{\phi \in X :\ \phi' \in X,\ \phi'(0) = M_0\phi(0) + \sum_{i=1}^m (M_i\phi(-\tau_i) + M_{-i}\phi(\tau_i))\},$$

$$\mathcal{L}_\xi \phi = \phi',\ \phi \in \mathcal{D}(\mathcal{L}_\xi)$$

Using Lagrange represenation of $P_n x$: $\mathcal{P}_N x = \sum_{k=-N}^{N} l_{N,k}\, x_k,$

$$\mathcal{L}_\xi^N = \begin{bmatrix} d_{-N,-N} & \dots & d_{-N,N} \\ \vdots & & \vdots \\ d_{-1,-N} & \dots & d_{-1,N} \\ a_{-N} & \dots & a_N \\ d_{1,-N} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{N,-N} & \dots & d_{N,N} \end{bmatrix} \in \mathbb{R}^{(2N+1)(2n)\times(2N+1)2n},$$

$$d_{i,k} = l'_{N,k}(\theta_{N,i})I, \text{ for } i,k \in \{-N,\dots,-1,1,\dots,N\},\ i \neq 0$$

$$\mathcal{L}_\xi \phi = \phi',\ \phi \in \mathcal{D}(\mathcal{L}_\xi)$$

These entries can be calculated beforehand.

$$\mathcal{L}_\xi^N = \begin{bmatrix} d_{-N,-N} & \dots & d_{-N,N} \\ \vdots & & \vdots \\ d_{-1,-N} & \dots & d_{-1,N} \\ a_{-N} & \dots & a_N \\ d_{1,-N} & \dots & d_{1,N} \\ \vdots & & \vdots \\ d_{N,-N} & \dots & d_{N,N} \end{bmatrix} \in \mathbb{R}^{(2N+1)(2n)\times(2N+1)2n},$$

$$\begin{aligned} a_0 &= M_0\, l_{N,0}(0) + \textstyle\sum_{k=1}^m (M_k l_{N,k}(-\tau_k) + M_{-k} l_{N,k}(\tau_k)) \\ a_k &= \textstyle\sum_{k=1}^m (M_k l_{N,k}(-\tau_k) + M_{-k} l_{N,k}(\tau_k)), \quad k \in \{-N, \dots, N\},\ k \neq 0. \end{aligned}$$

$$\mathcal{D}(\mathcal{L}_\xi) = \{\phi \in X :\ \phi' \in X,\ \phi'(0) = M_0\phi(0) + \sum_{i=1}^m (M_i\phi(-\tau_i) + M_{-i}\phi(\tau_i))\},$$

- The eigenvalue problem for L_ξ^N can be written as a sparse generalized eigenvalue problem (large-scale methods)
- [Prop 2.1] symmetric eigenvalues with respect to the imaginary axis if

$$\theta_{N,-i} = -\theta_{N,i}, \ i = 1, \ldots, N,$$

- We are interested in the imaginary axis eigenvalues of L_ξ typically among the smallest eigenvalues)

$$\sigma_i(G(j\omega_0) = \xi \iff \det(j\omega_0 I - L_\xi) = 0$$

- A small value of N is sufficient in most practical problems for computing a good approximation of the H_∞-norm for correction step

$$\sigma_i(G(j\omega_0) = \xi \leftrightsquigarrow \det(j\hat{\omega}_0 I - L_\xi^N) = 0$$

Correcting the H_∞ Norm

We want to correct the approximate results from prediction step.

Before that:

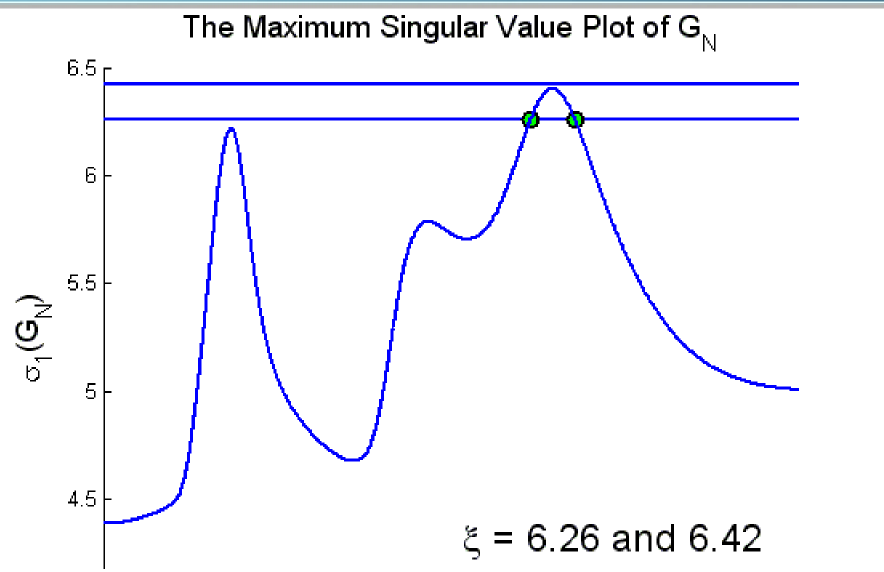
[Thm 4.1] Let ζ> 0 be such that the matrix $D_\xi := D^T D - \xi^2 I$ is non-singular. λ is an eigenvalue of L_ζ if and only if

$$h_\xi(\lambda) := \det H_\xi(\lambda) = 0$$

where

$$H_\xi(\lambda) := \lambda I - M_0 - \sum_{i=1}^{m} \left(M_i e^{-\lambda\tau_i} + M_{-i} e^{\lambda\tau_i} \right)$$

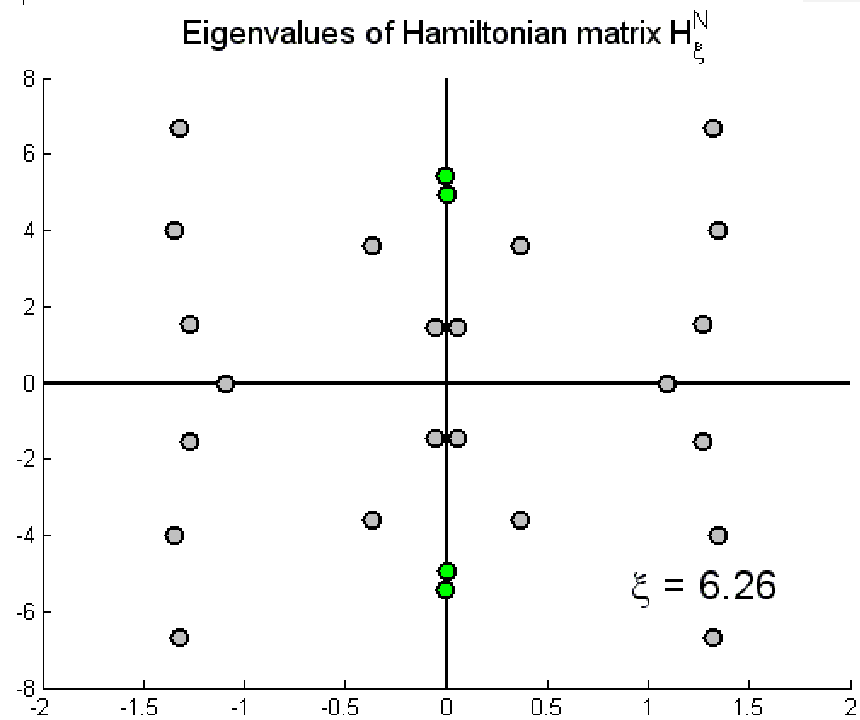
$\sigma_i(G(j\omega_0)) = \xi \iff$ Thm 2.1	$L_\xi u = j\omega_0 u$	$\iff \det H_\xi(j\omega_0)$ Thm 4.1

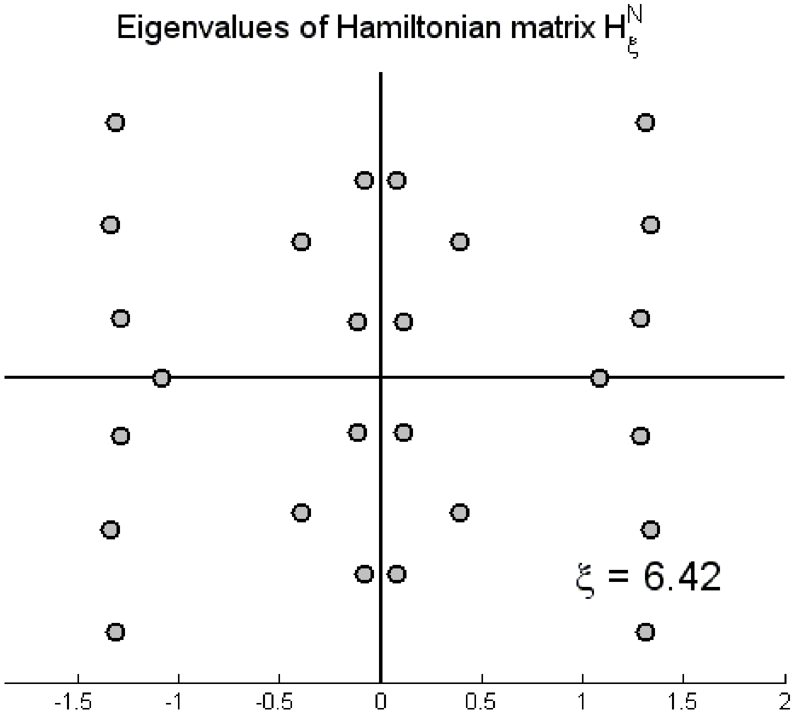


If $\|G(j\omega)\|_{\mathcal{H}_\infty} = \hat{\xi} = \sigma_1(G(j\hat{\omega}))$

then $(\hat{\omega}, \hat{\xi})$ satisfies

$$h_\xi(j\omega) = 0, \quad h'_\xi(j\omega) = 0$$





Since $h_\xi(j\omega) = 0, \quad h'_\xi(j\omega) = 0$

Using the properties of $h_\xi(j\omega)$,

$$\begin{cases} H(j\omega,\ \xi) \begin{bmatrix} u, \\ v \end{bmatrix} = 0, \quad n(u,v) = 0 \\ \Im\left\{v^* \left(I + \sum_{i=1}^{p} A_i \tau_i e^{-j\omega\tau_i}\right) u\right\} = 0 \end{cases}$$

- Overdetermined system (4n+3 equations, 4n+2 unknowns)
- It can be solved least-square sense via optimization

Main Idea in a nutshell

Prediction step

Calculate the approximate H_∞ norm as

$$\|G\|_{\mathcal{H}_\infty} \approx \sup\{\xi \in \mathbb{R}_+ : \text{ matrix } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$$

Correction step

correct the approximate results from the prediction step

Main Idea in a nutshell

Prediction step

For fixed N, determine

$$\sup\{\xi \in \mathbb{R}_+ : \text{ matrix } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$$

and determine the corresponding eigenvalues on the imaginary axis

Correction step

Correct the results by solving the equations

$$\begin{cases} H(j\omega,\ \xi)\begin{bmatrix} u, \\ v \end{bmatrix} = 0, \quad n(u,v) = 0 \\ \Im\left\{v^*\left(I + \sum_{i=1}^{p} A_i \tau_i e^{-j\omega\tau_i}\right) u\right\} = 0 \end{cases}$$

Interpretating the Discretization of L_ζ

$$\sigma_i(G(j\omega_0) = \xi \;\leftrightsquigarrow\; \det(j\hat{\omega}_0 I - L_\xi^N) = 0$$

$$\sigma_i(?) = \xi \;\Longleftrightarrow\; \det(j\hat{\omega}_0 I - L_\xi^N) = 0$$

[Thm 5.1] Assume that $-\tau_{\max} \leq \theta_{N,-N} < \ldots < \theta_{N,0} = 0 < \cdots < \theta_{N,N} \leq \tau_{\max}$ is symmetric. Let p_N be the polynomial of the degree 2N+1 satisfying the conditions,

$$\begin{aligned} p_N(0;\ \lambda) &= 1, \\ p_N'(\theta_i;\lambda) &= \lambda p_N(\theta_i;\lambda),\ i = -N,\ldots,-1,1,\ldots,N. \end{aligned}$$

Let ζ> 0 be such that the matrix $D_\xi := D^T D - \xi^2 I$ is nonsingular. The matrix L_ζ^N has an imaginary axis eigenvalue if and only if $G_N(j\omega)$ has a singular value equal to ξ where

$$G_N(j\omega) = C\left(j\omega I - A_0 - \sum_{i=1}^{m} A_i p_N(-\tau_i;\ j\omega)\right)^{-1} B + D.$$

$$\sigma_i(G_N(j\omega_0)) = \xi \iff \det(j\omega_0 I - L_\xi^N) = 0$$

- It guarantees that L_ζ^N has imaginary axis eigenvalues for

$$\xi \in [\sigma_1(D),\ \|G_N(j\omega)\|_{\mathcal{H}_\infty}]$$

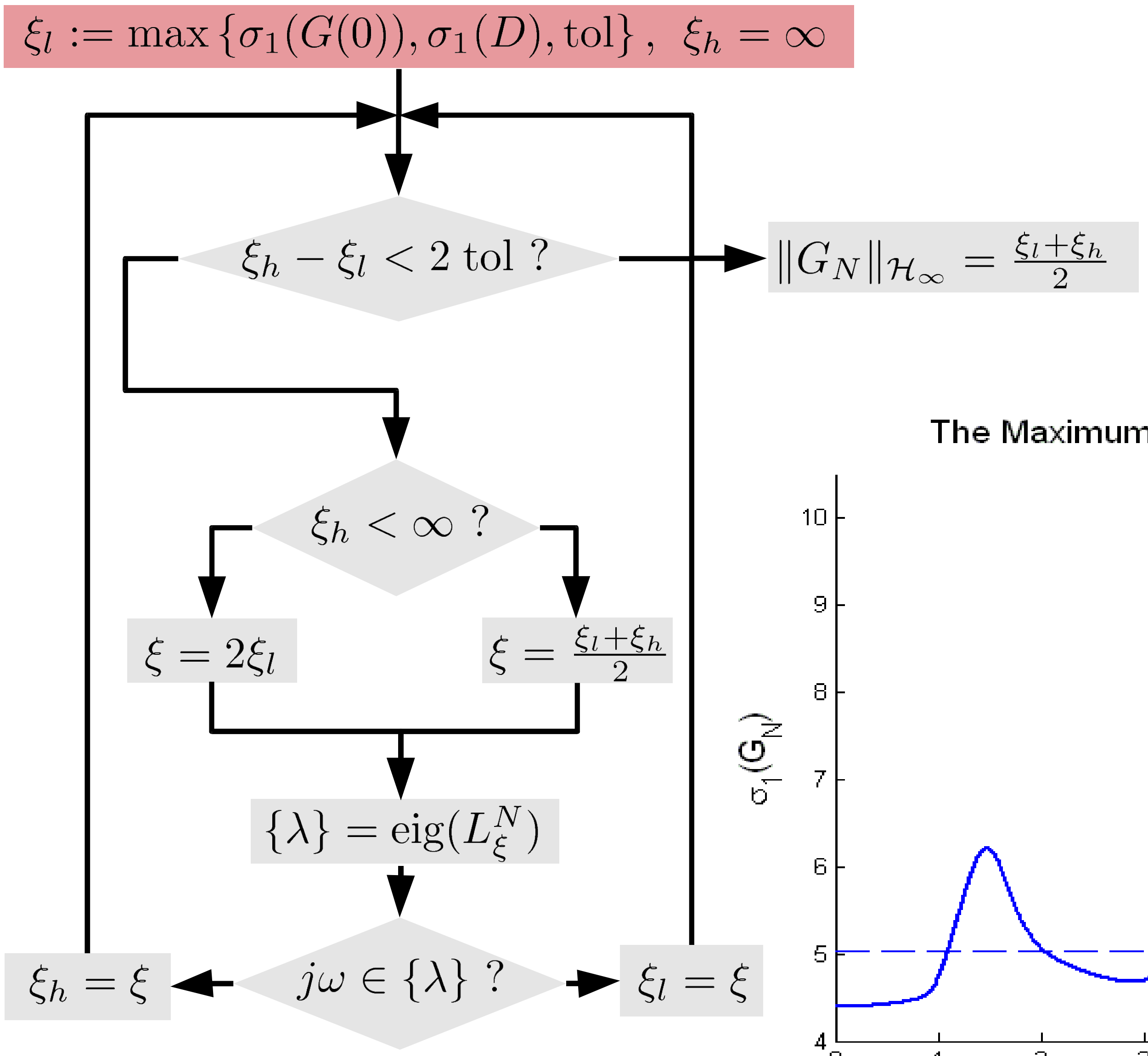
- No imaginary axis eigenvalues of L_ζ^N for

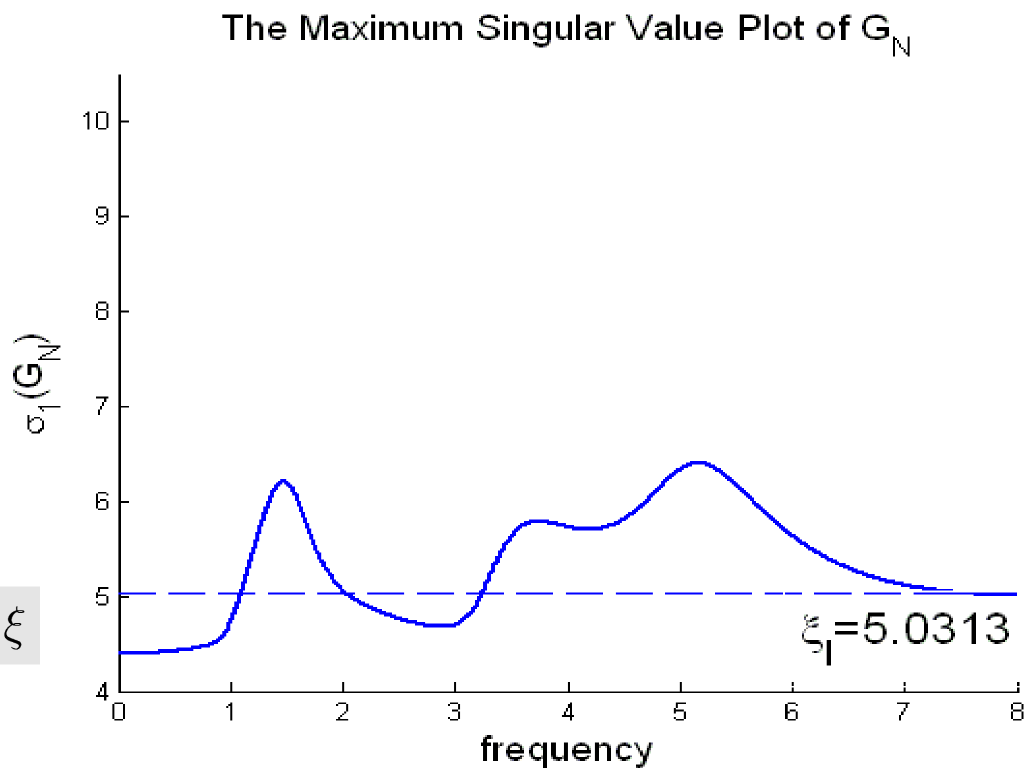
$$\xi > \|G_N(j\omega)\|_{\mathcal{H}_\infty}$$

- Thus the supremum exists

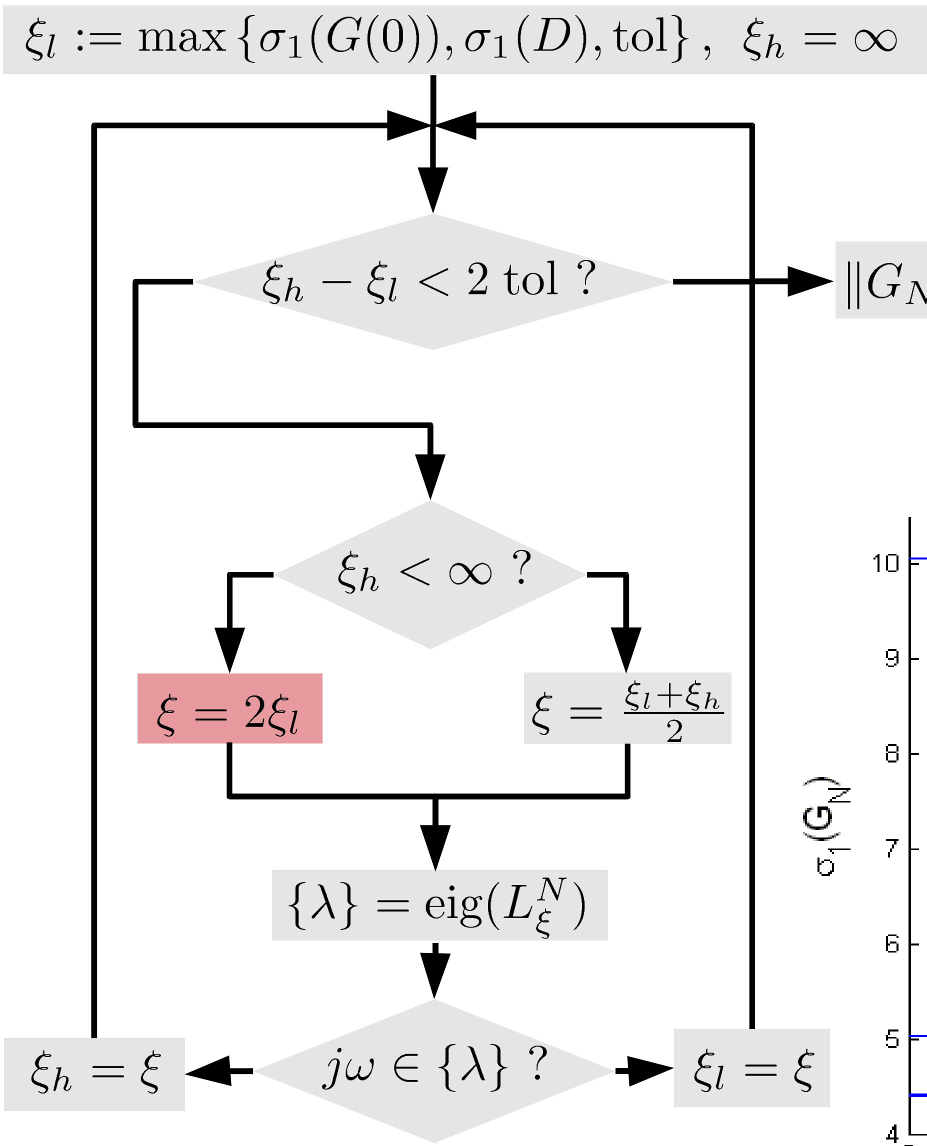
$$\sup\{\xi \in \mathbb{R}_+ :\ \text{matrix } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$$

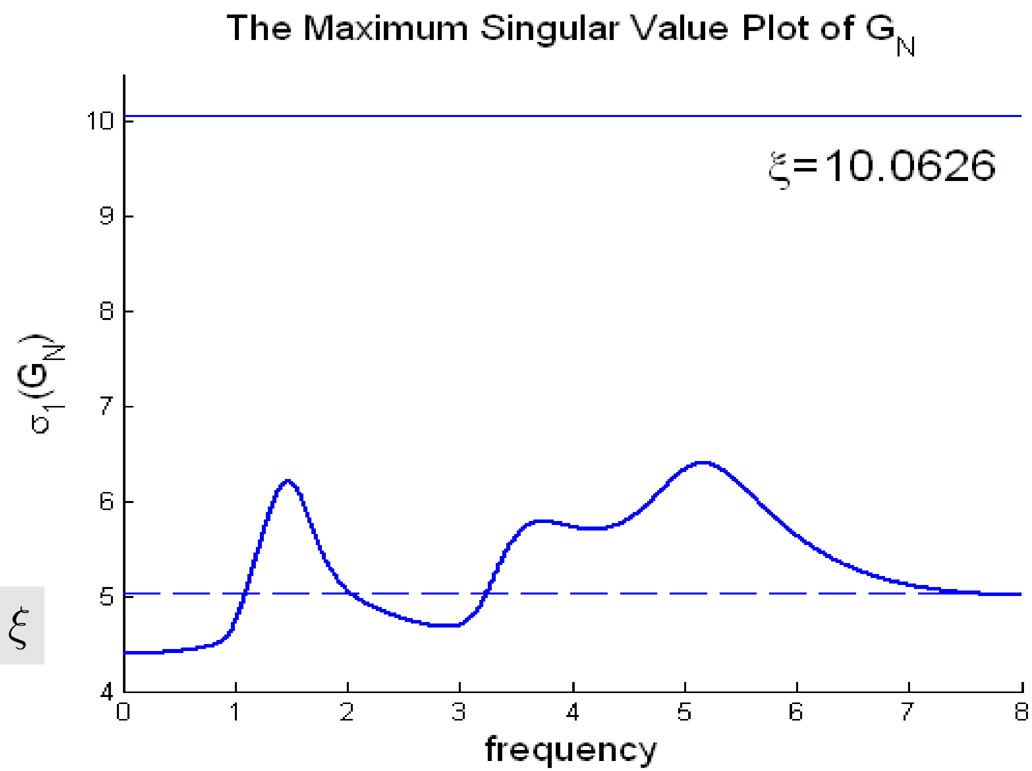
First Algorithm for H_∞ Norm Computation for TDS – Prediction Step



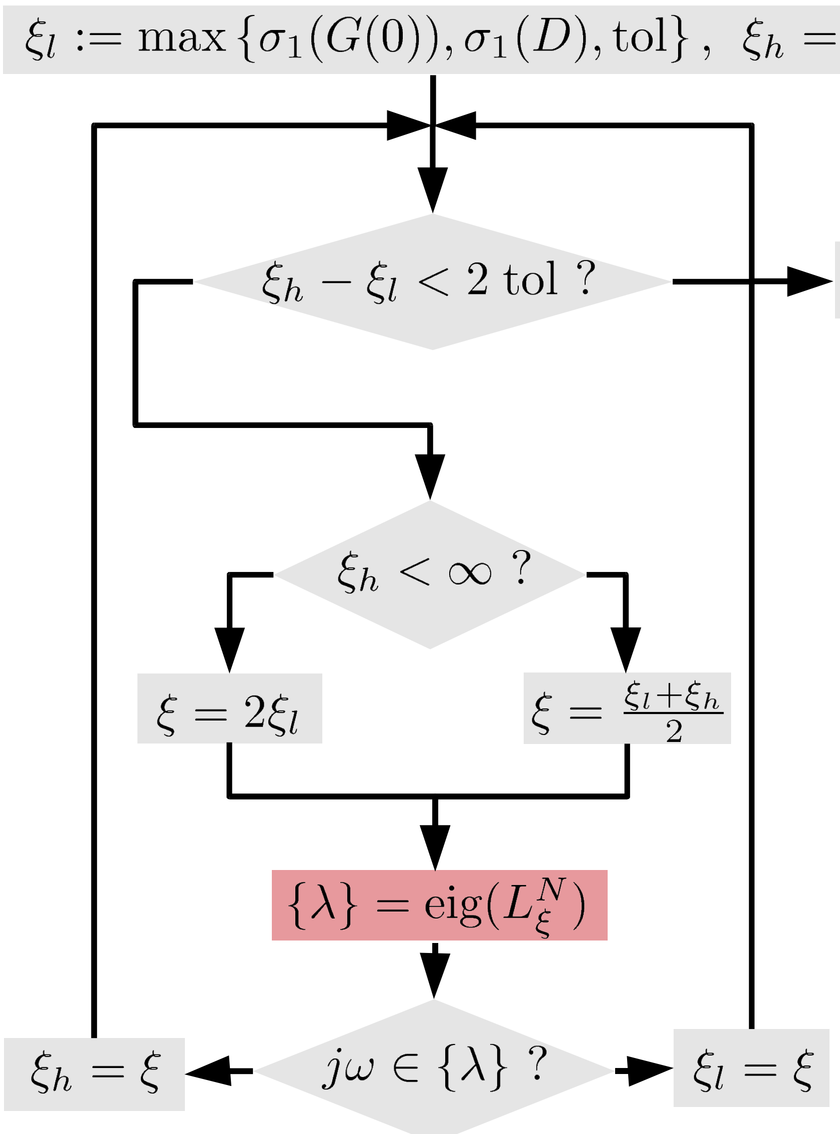


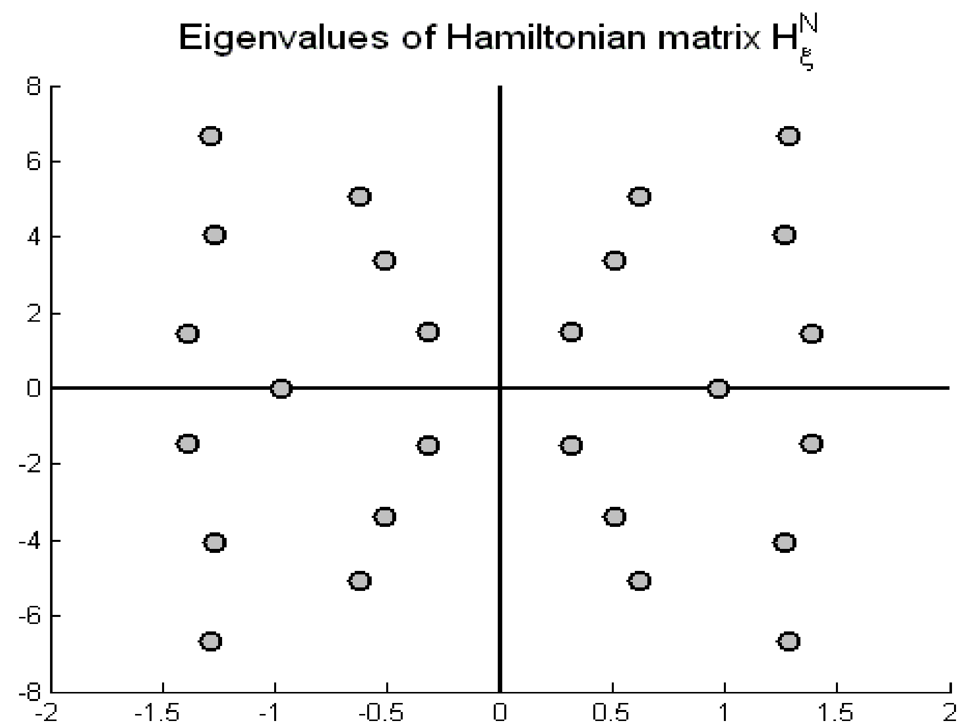
First Algorithm for H_∞ Norm Computation for TDS – Prediction Step



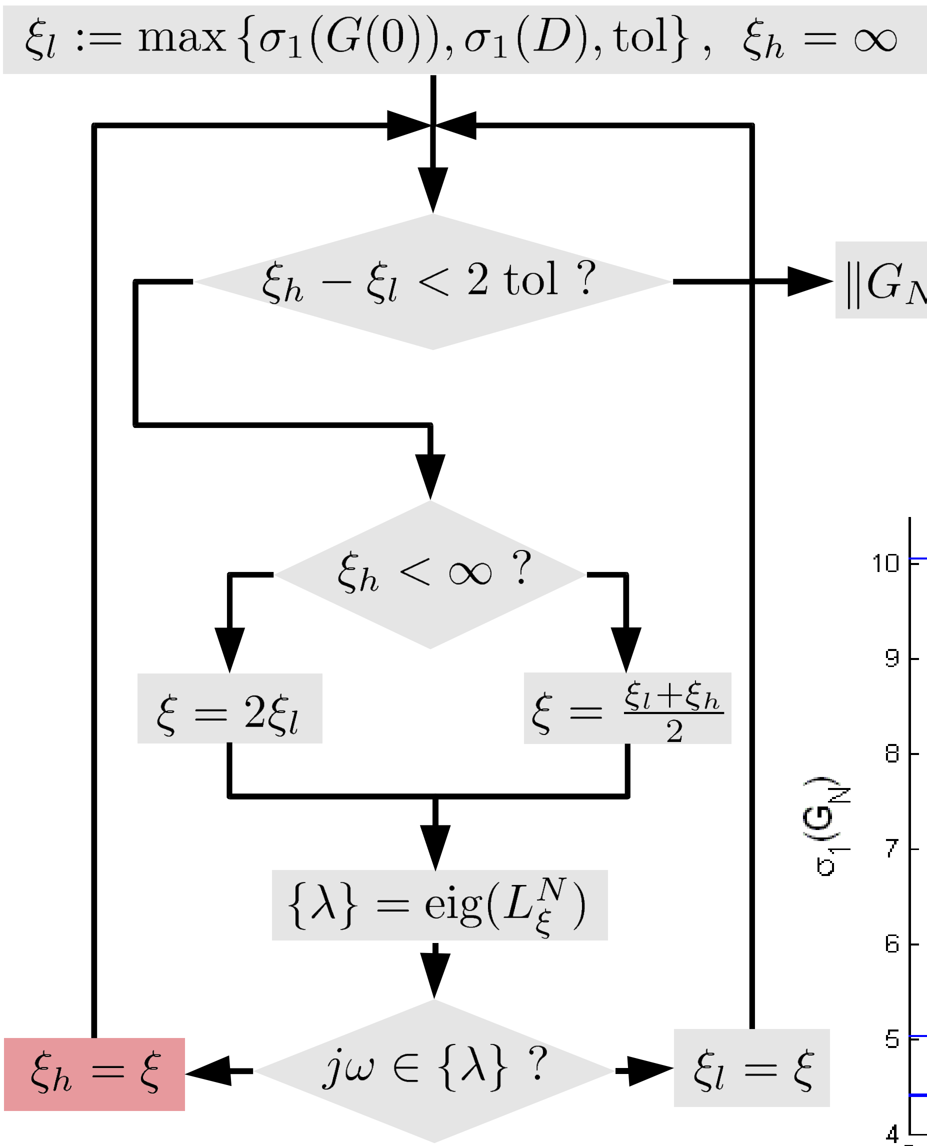


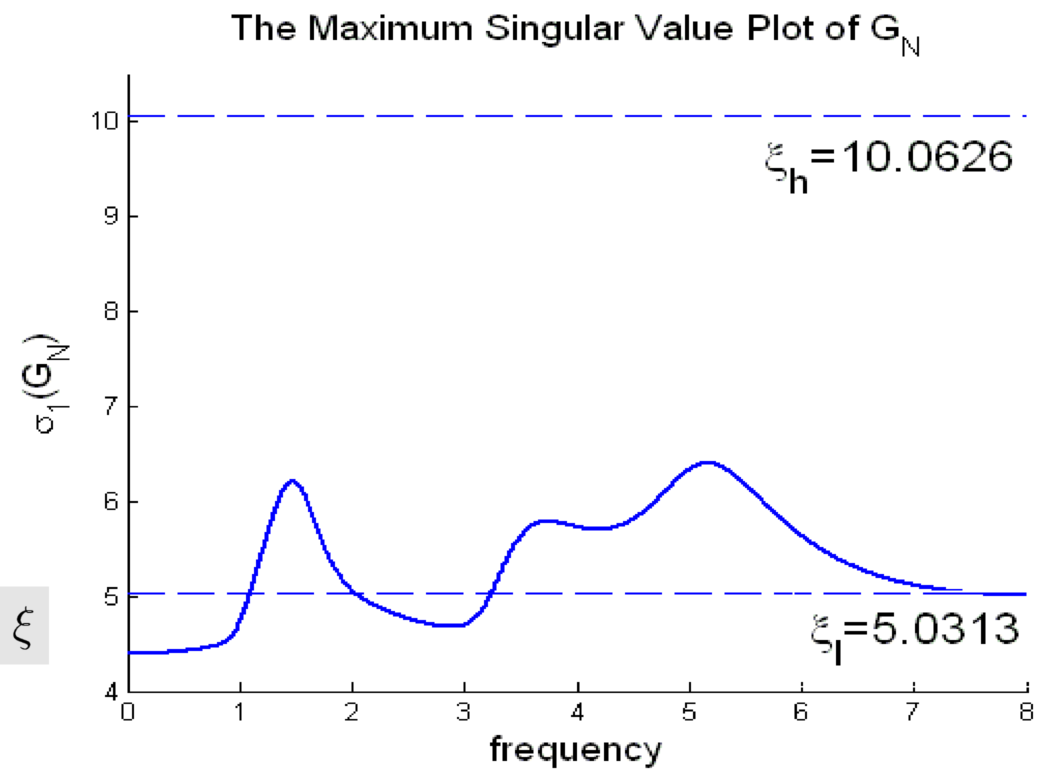
First Algorithm for H_∞ Norm Computation for TDS – Prediction Step



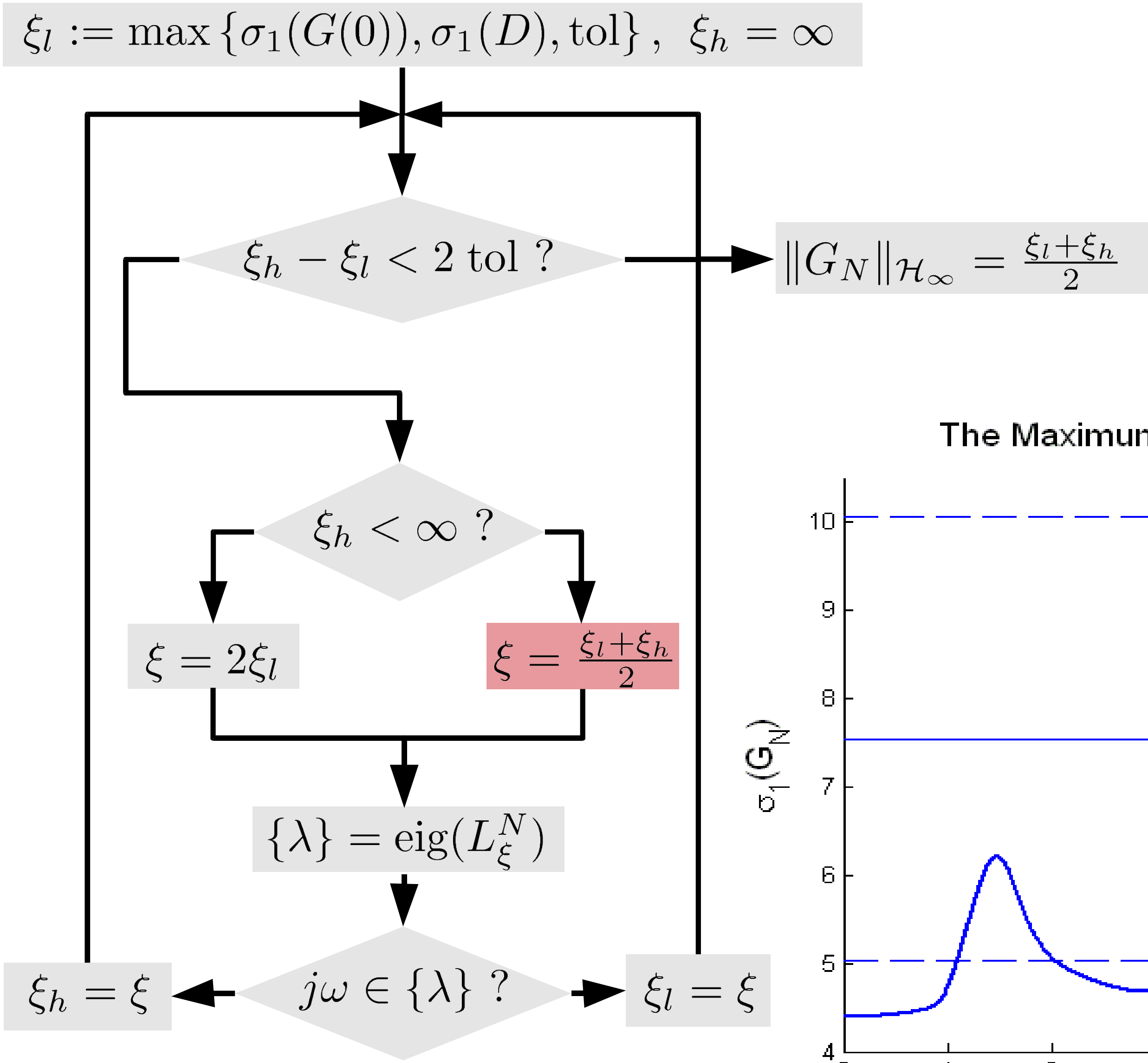


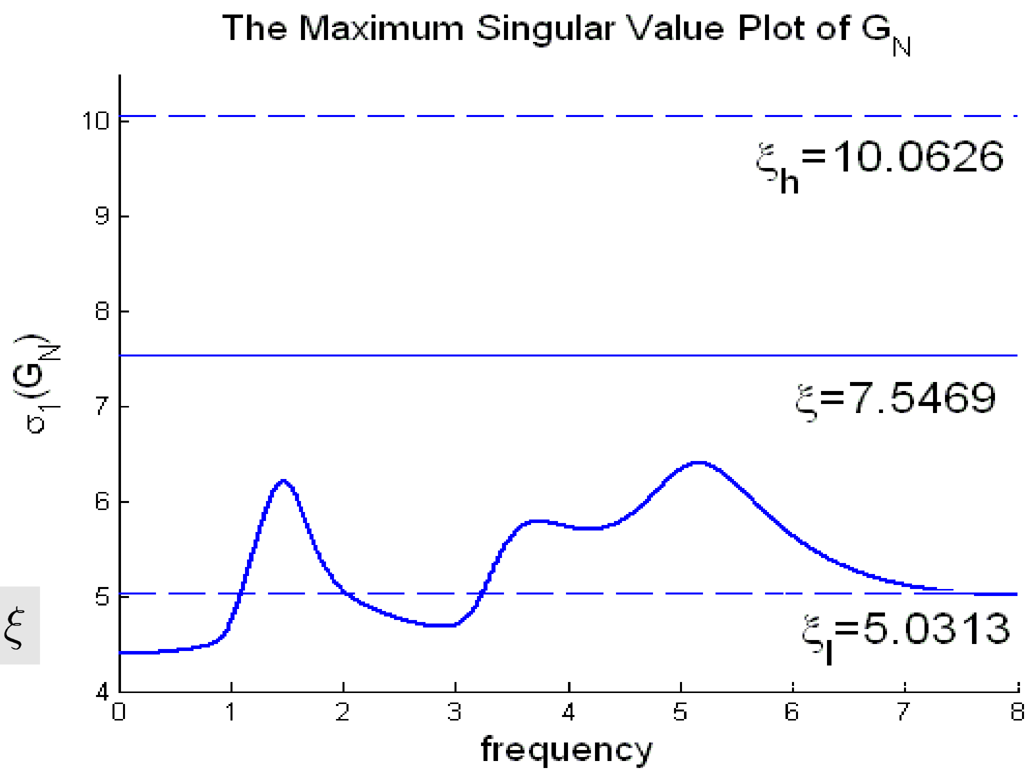
First Algorithm for H_∞ Norm Computation for TDS – Prediction Step

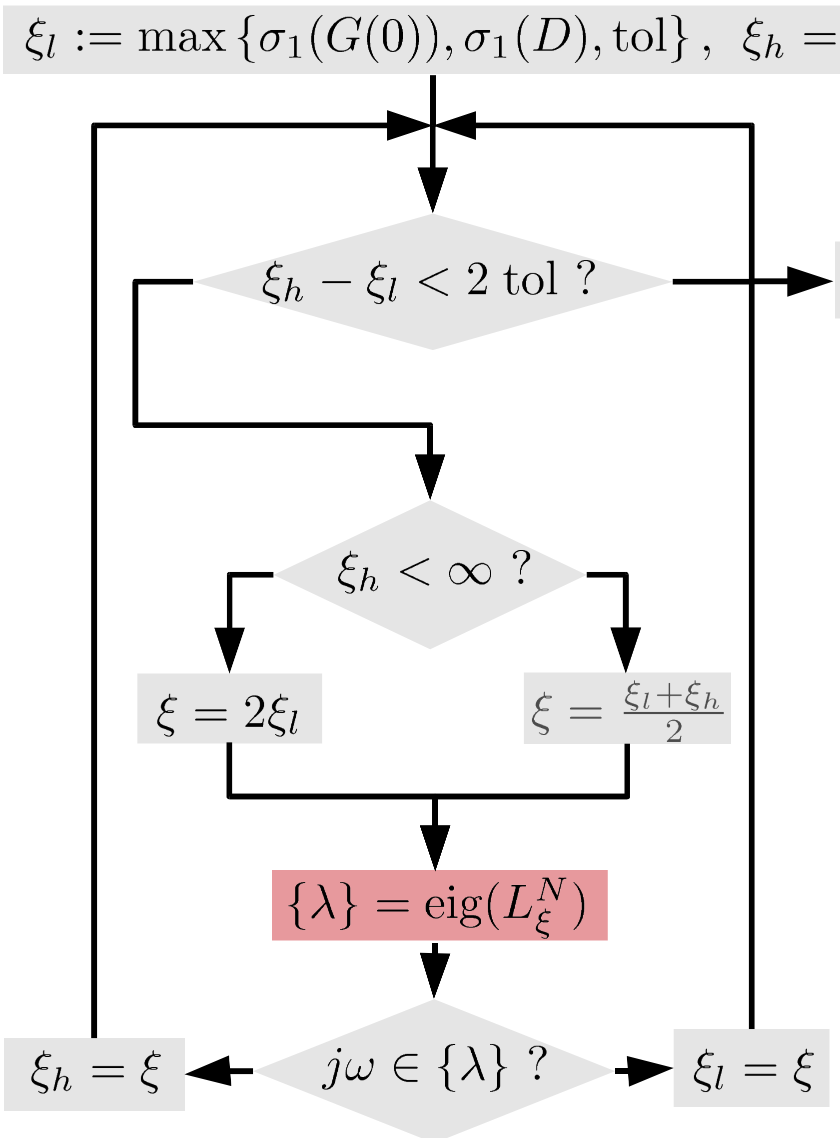




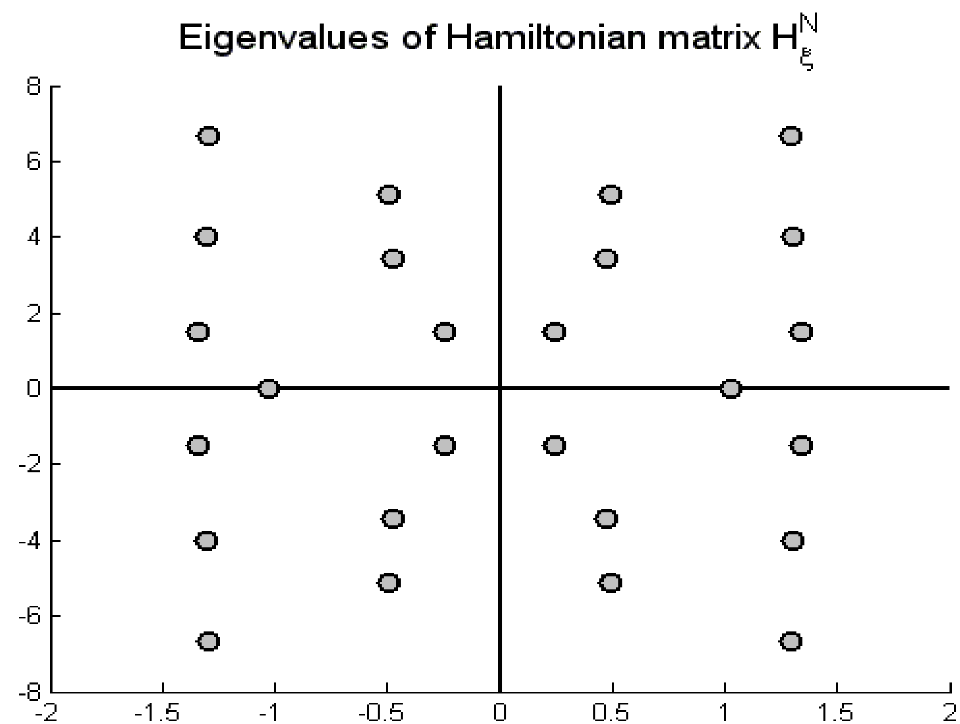
First Algorithm for H_∞ Norm Computation for TDS – Prediction Step





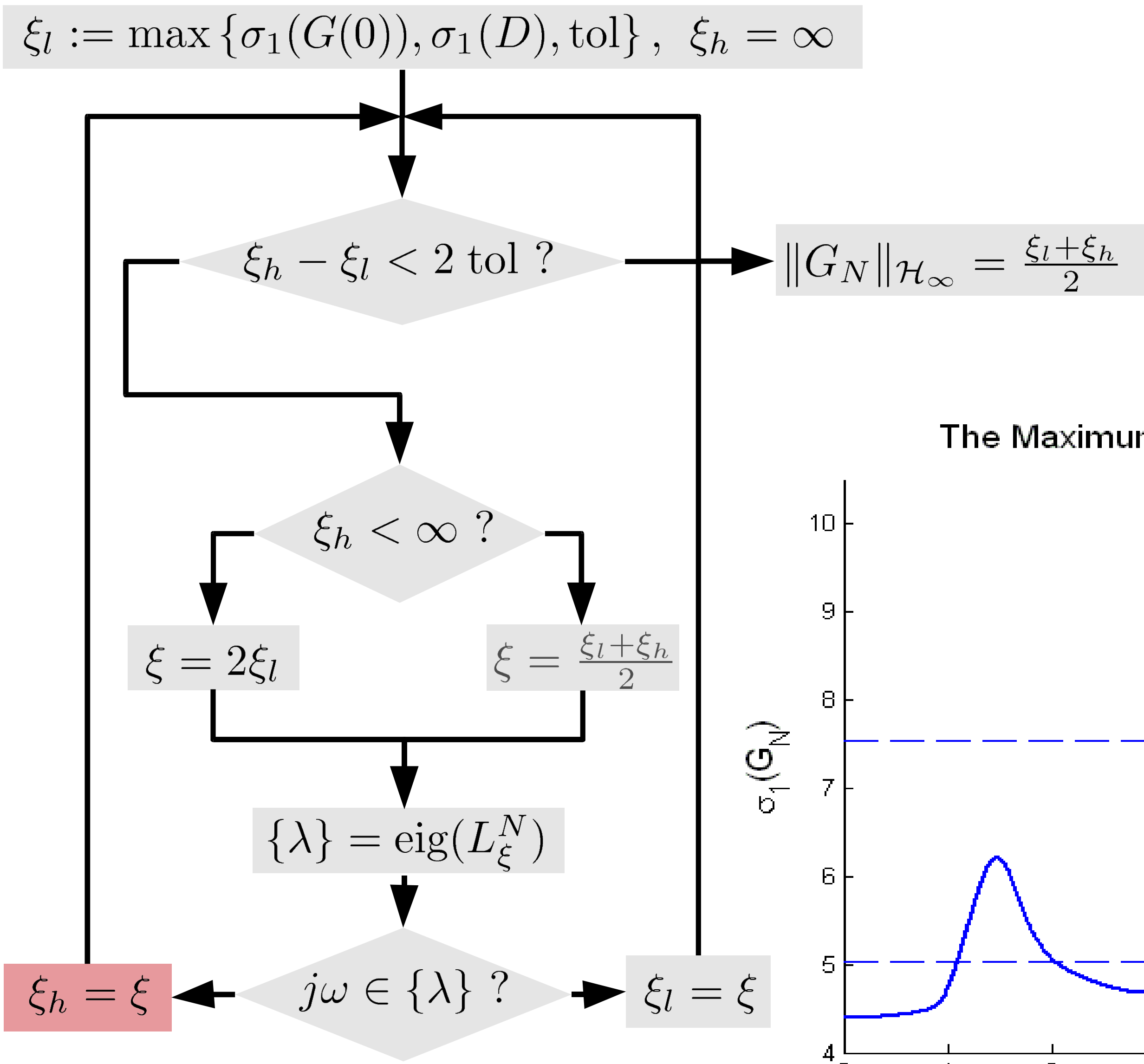


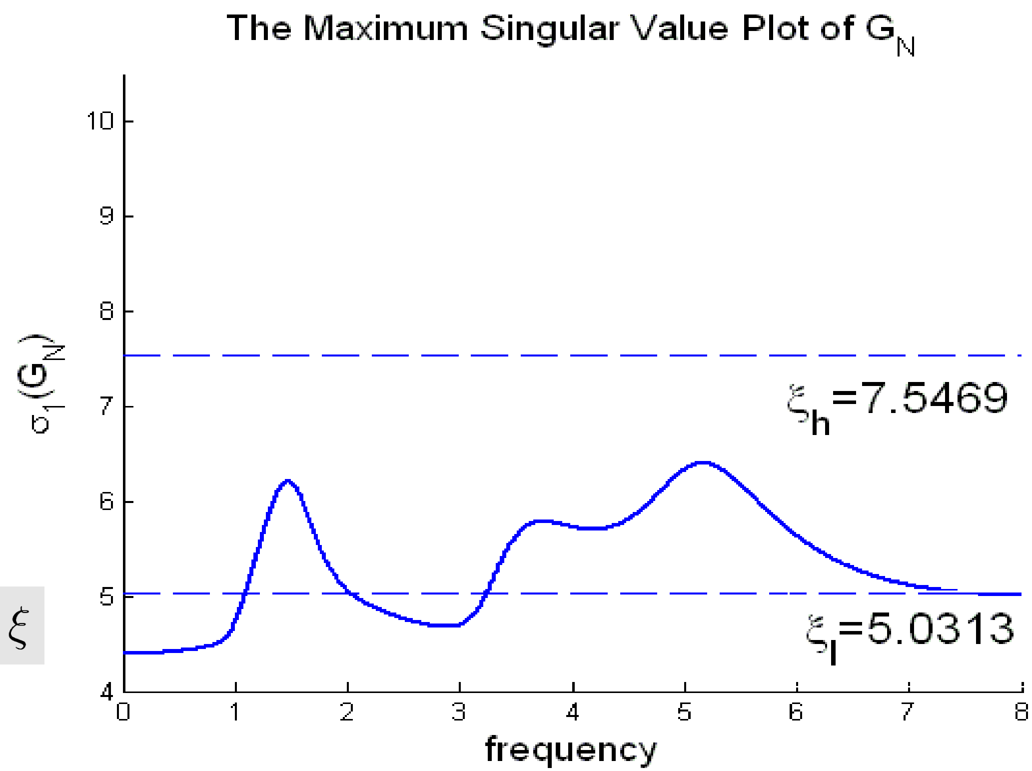
$\xi_l := \max\{\sigma_1(G(0)), \sigma_1(D), \text{tol}\}, \quad \xi_h = \infty$
$\xi_h - \xi_l < 2\ \text{tol}$?
$\|G_N\|_{\mathcal{H}_\infty} = \frac{\xi_l+\xi_h}{2}$
$\xi_h < \infty$?
$\xi = 2\xi_l$
$\xi = \frac{\xi_l+\xi_h}{2}$
$\{\lambda\} = \text{eig}(L_\xi^N)$
$j\omega \in \{\lambda\}$?
$\xi_h = \xi$
$\xi_l = \xi$



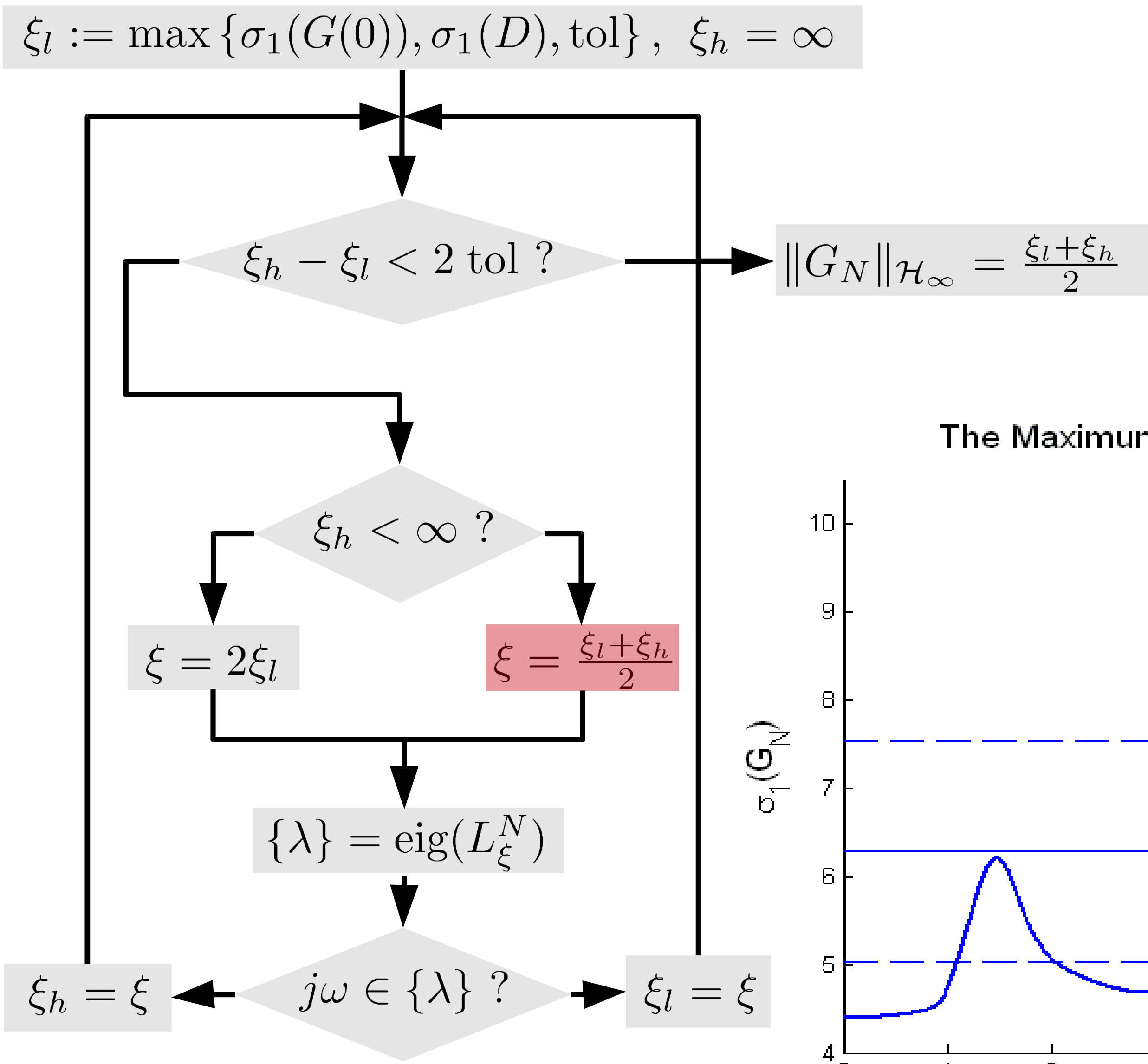
Eigenvalues of Hamiltonian matrix H_ξ^N
8
6
4
2
0
-2
-4
-6
-8
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

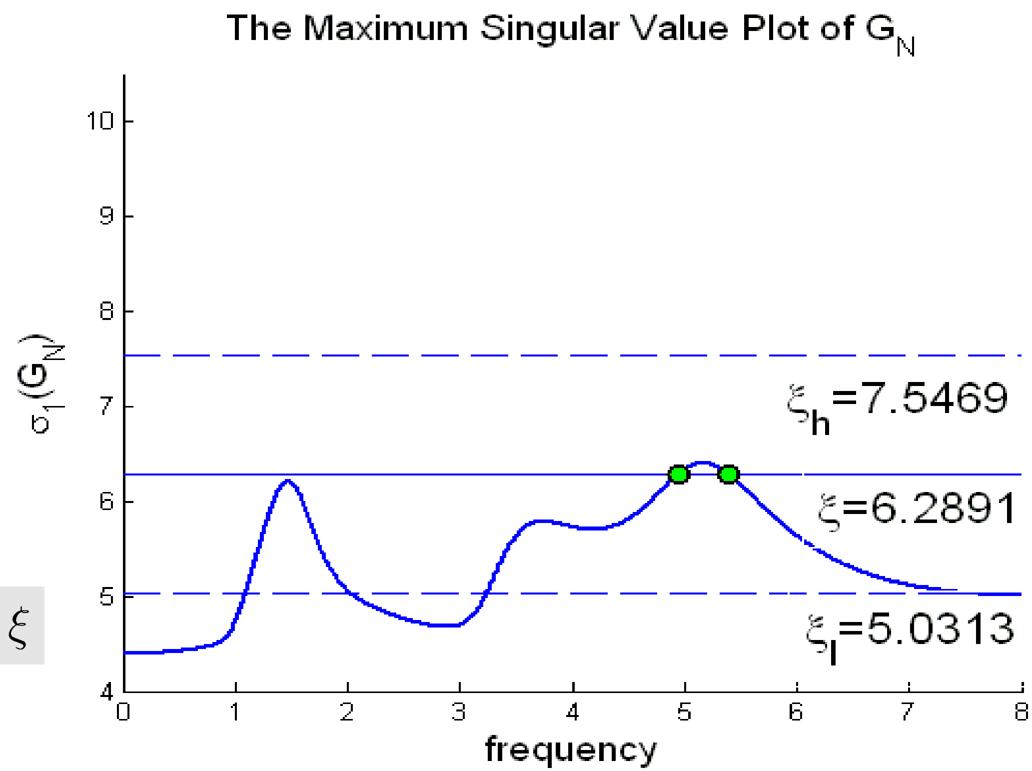
First Algorithm for H_∞ Norm Computation for TDS – Prediction Step



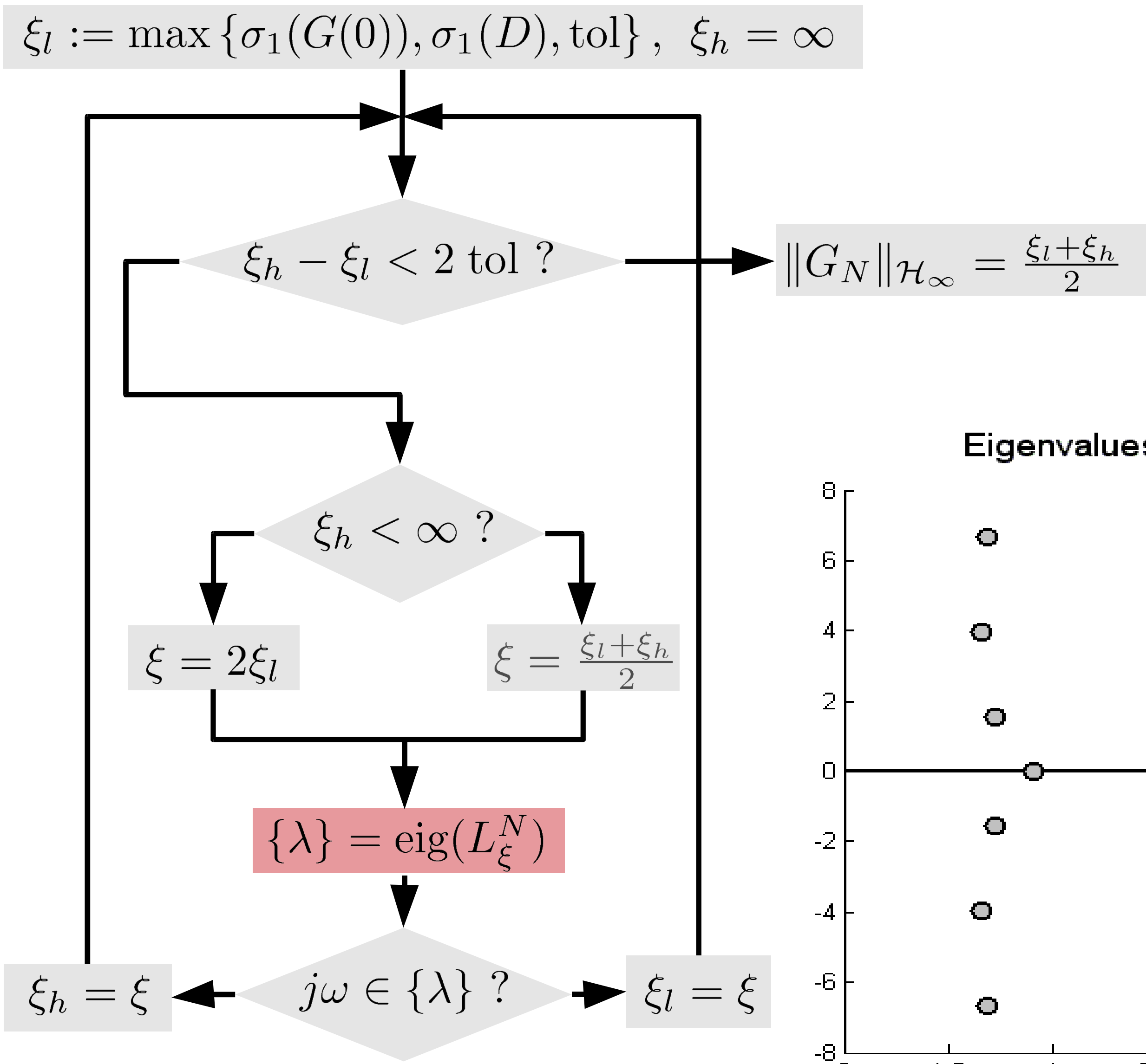


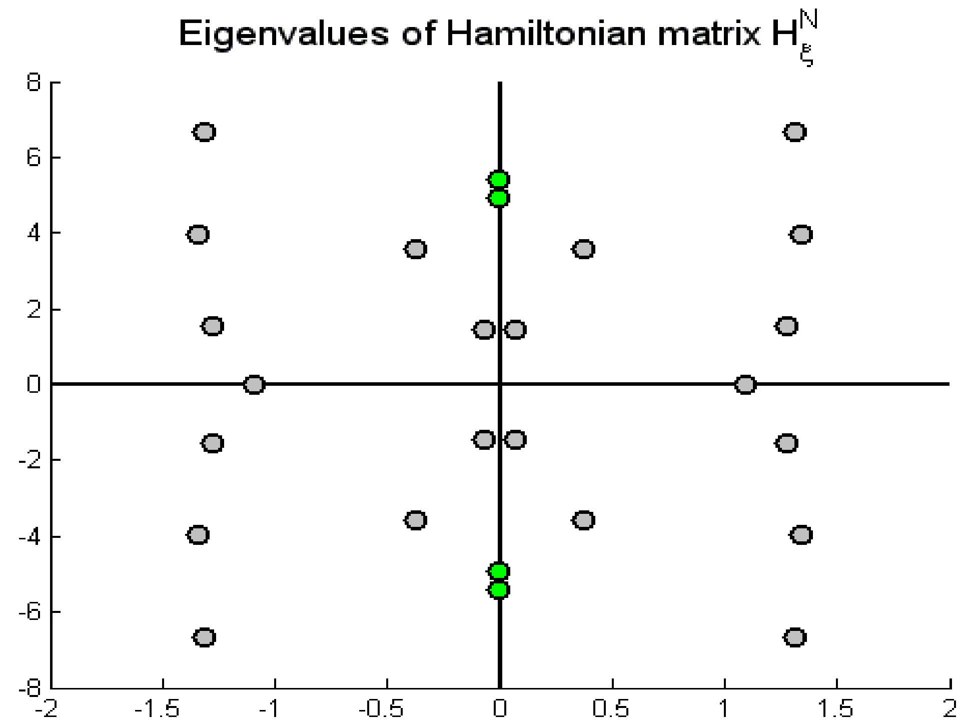
First Algorithm for H_∞ Norm Computation for TDS – Prediction Step



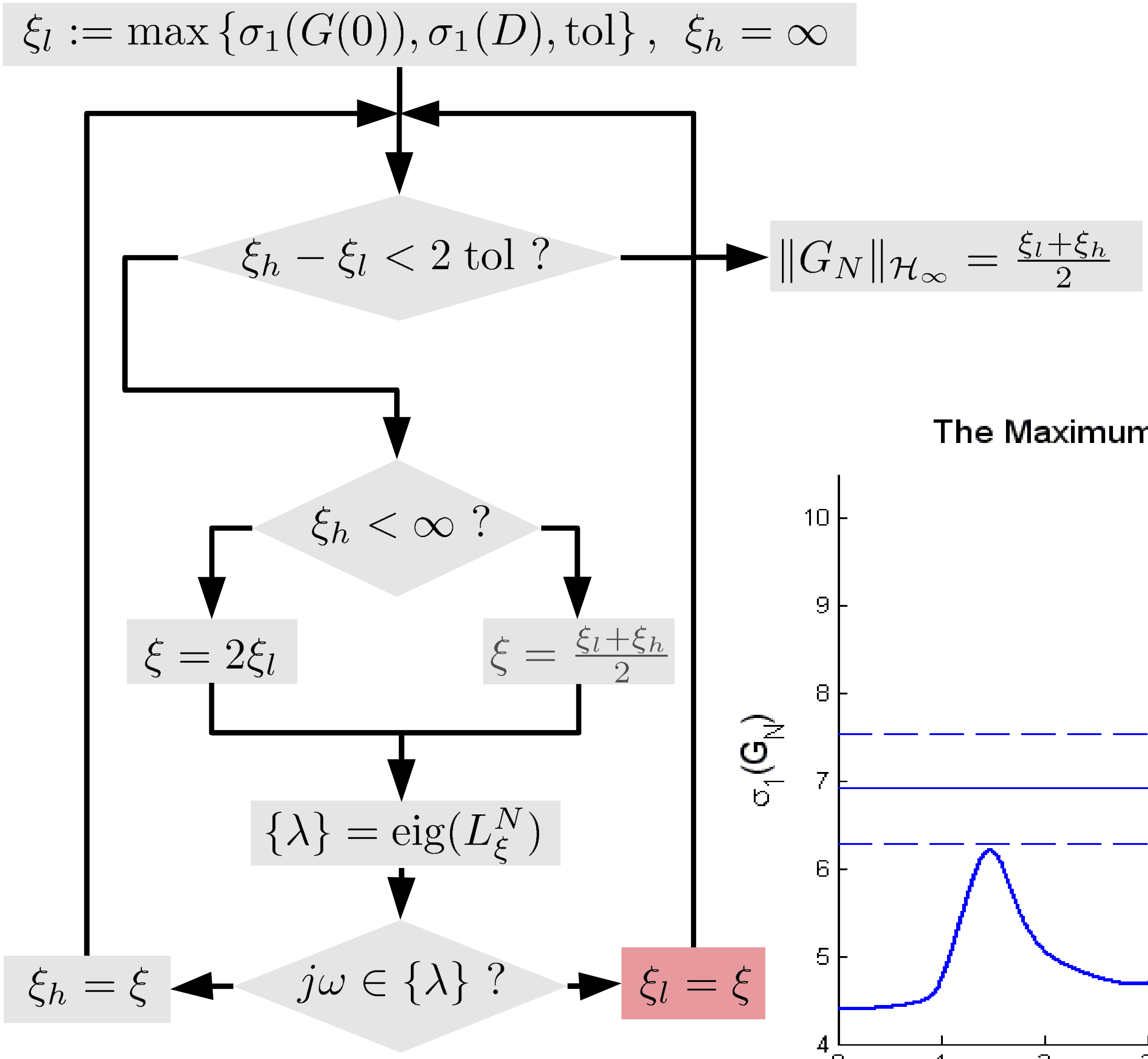


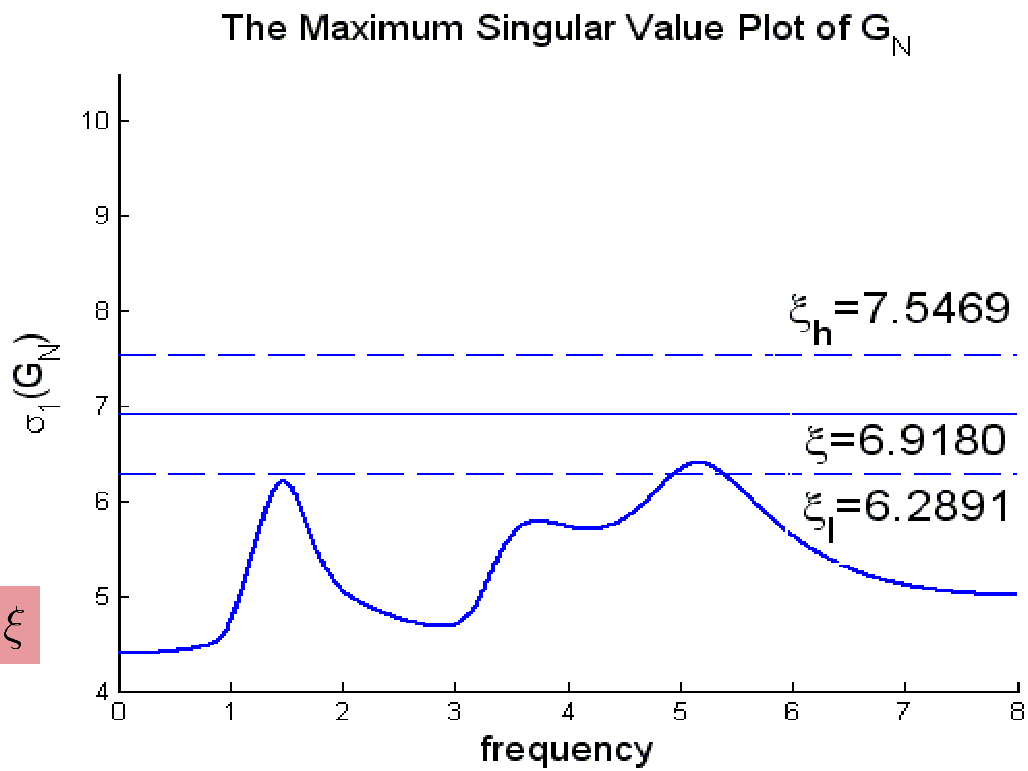
First Algorithm for H_∞ Norm Computation for TDS – Prediction Step

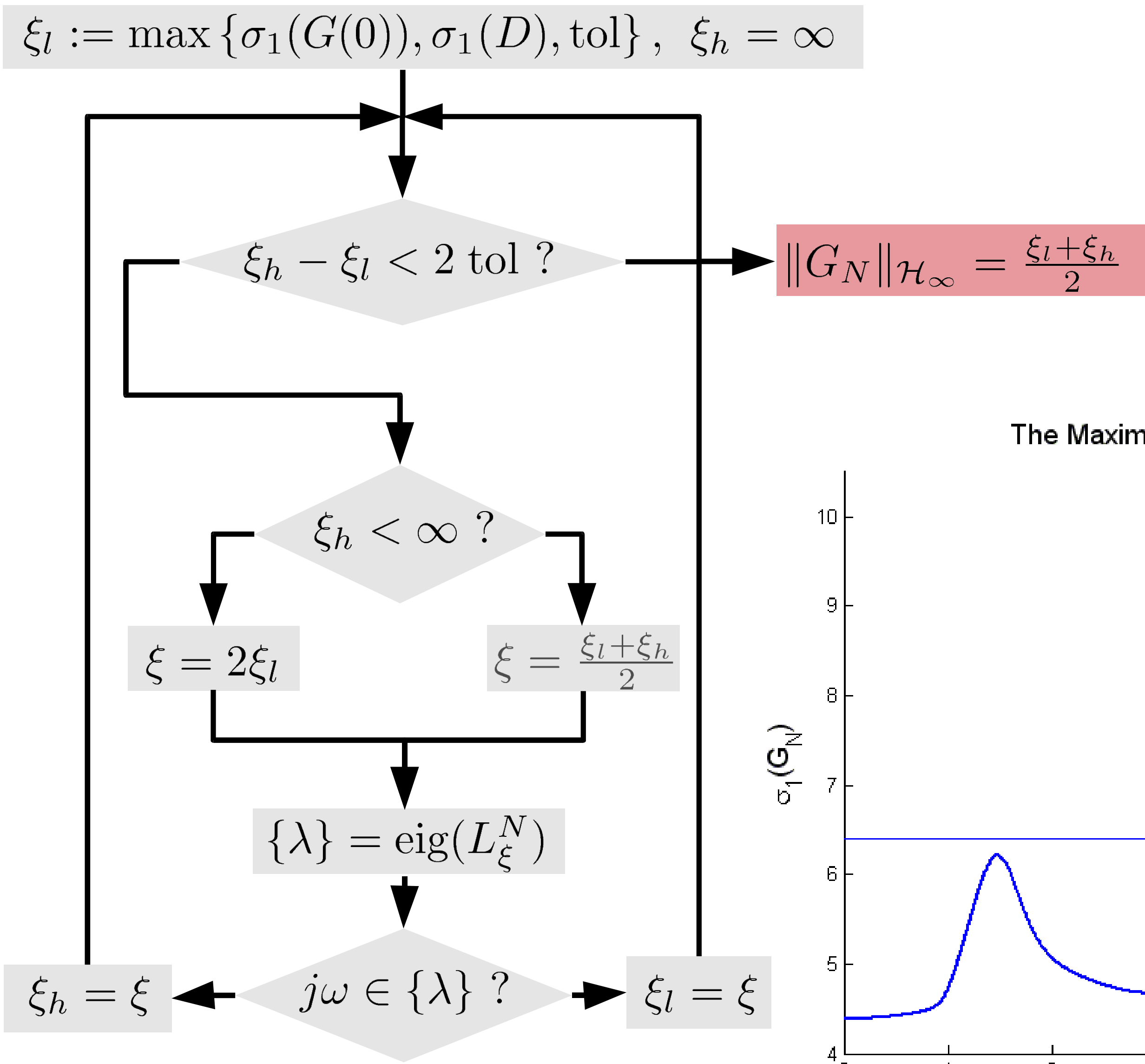


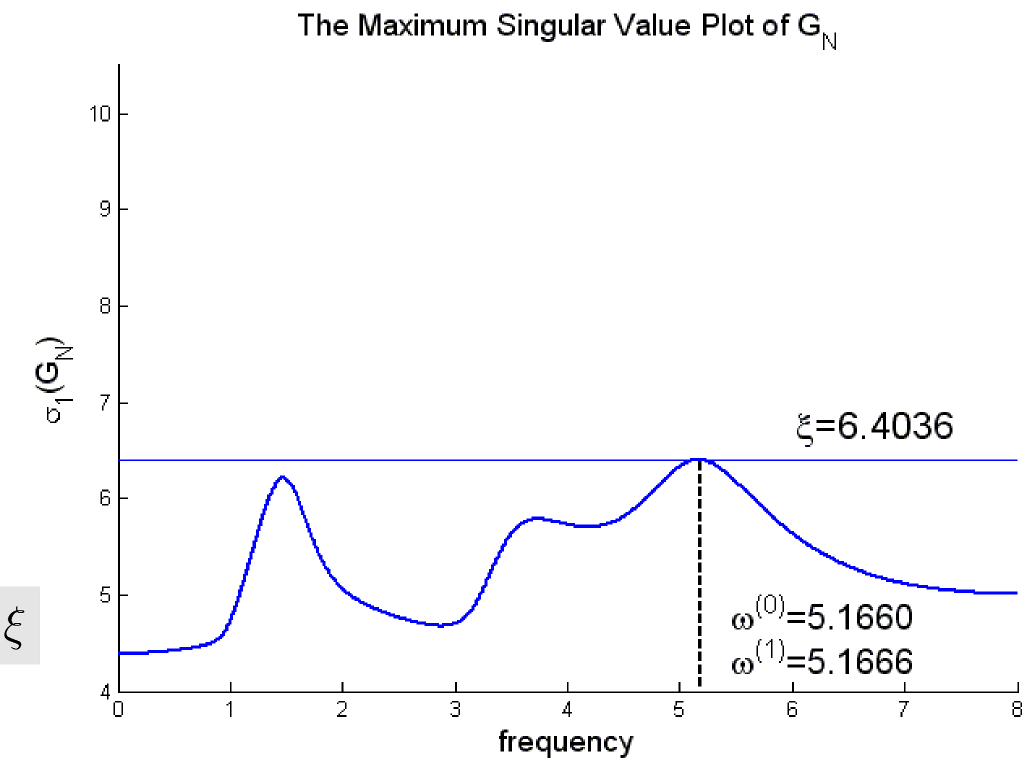


First Algorithm for H_∞ Norm Computation for TDS – Prediction Step






$\xi_l := \max\{\sigma_1(G(0)), \sigma_1(D), \text{tol}\}, \quad \xi_h = \infty$
$\xi_h - \xi_l < 2\ \text{tol}$?
$\|G_N\|_{\mathcal{H}_\infty} = \frac{\xi_l+\xi_h}{2}$
$\xi_h < \infty$?
$\xi = 2\xi_l$
$\xi = \frac{\xi_l+\xi_h}{2}$
$\{\lambda\} = \text{eig}(L_\xi^N)$
$j\omega \in \{\lambda\}$?
$\xi_h = \xi$
$\xi_l = \xi$


The Maximum Singular Value Plot of G_N
$\sigma_1(G_N)$
frequency
ξ=6.4036
$\omega^{(0)}$=5.1660
$\omega^{(1)}$=5.1666

First Algorithm for H_∞ Norm Computation for TDS – Correction Step

- Determine all eigenvalues $\{j\omega^{(1)},\ldots,j\omega^{(p)}\}$ of L_ζ^N on the positive imaginary axis, and the corresponding eigenvectors $\{x^{(1)},\ldots,x^{(p)}\}$

- For all i=1,...,p solve

$$\begin{cases} H(j\omega,\ \xi)\begin{bmatrix} u, \\ v \end{bmatrix} = 0, \quad n(u,v)=0 \\ \Im\left\{v^*\left(I+\sum_{i=1}^{p} A_i\tau_i e^{-j\omega\tau_i}\right)u\right\}=0 \end{cases}$$

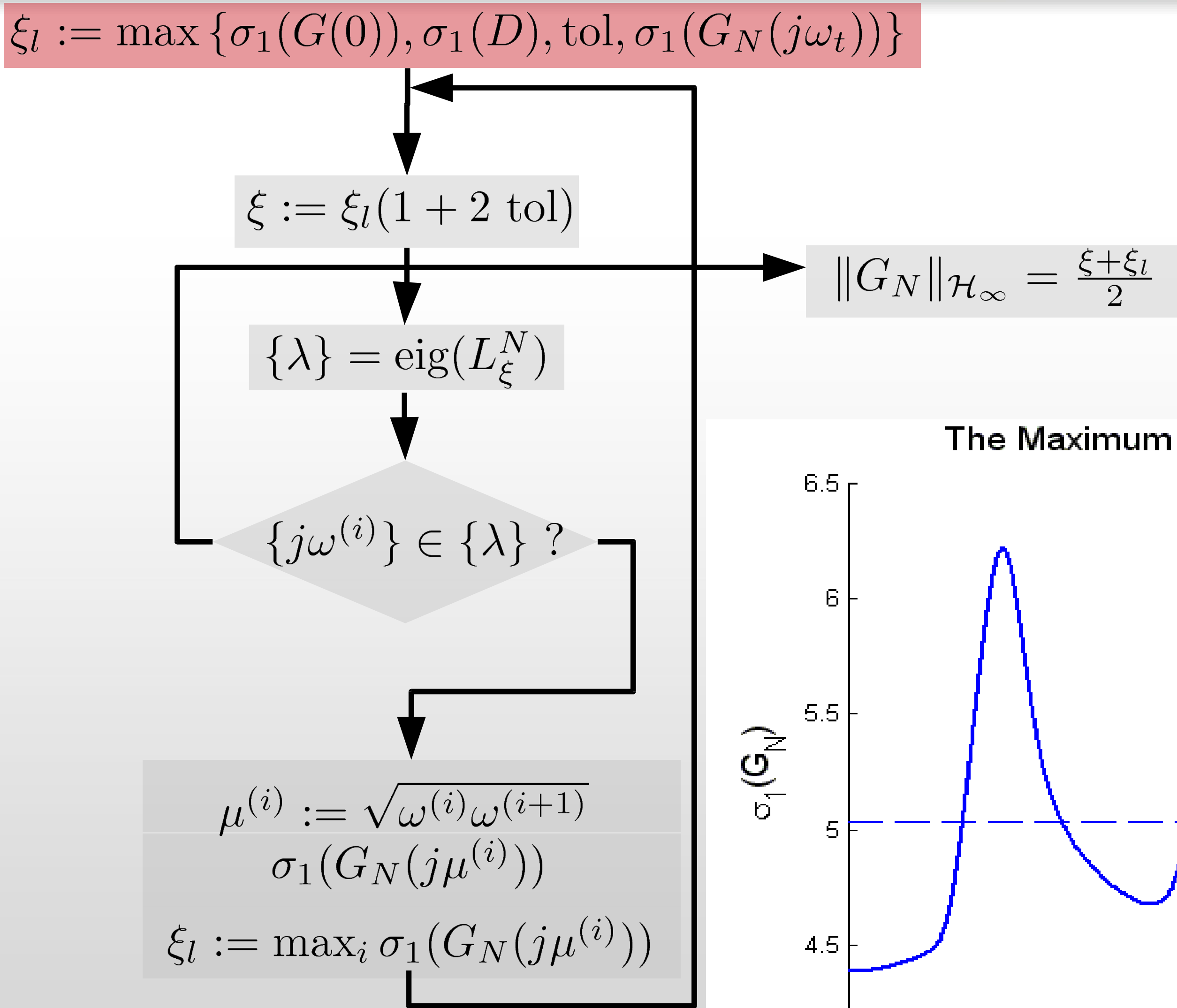
where

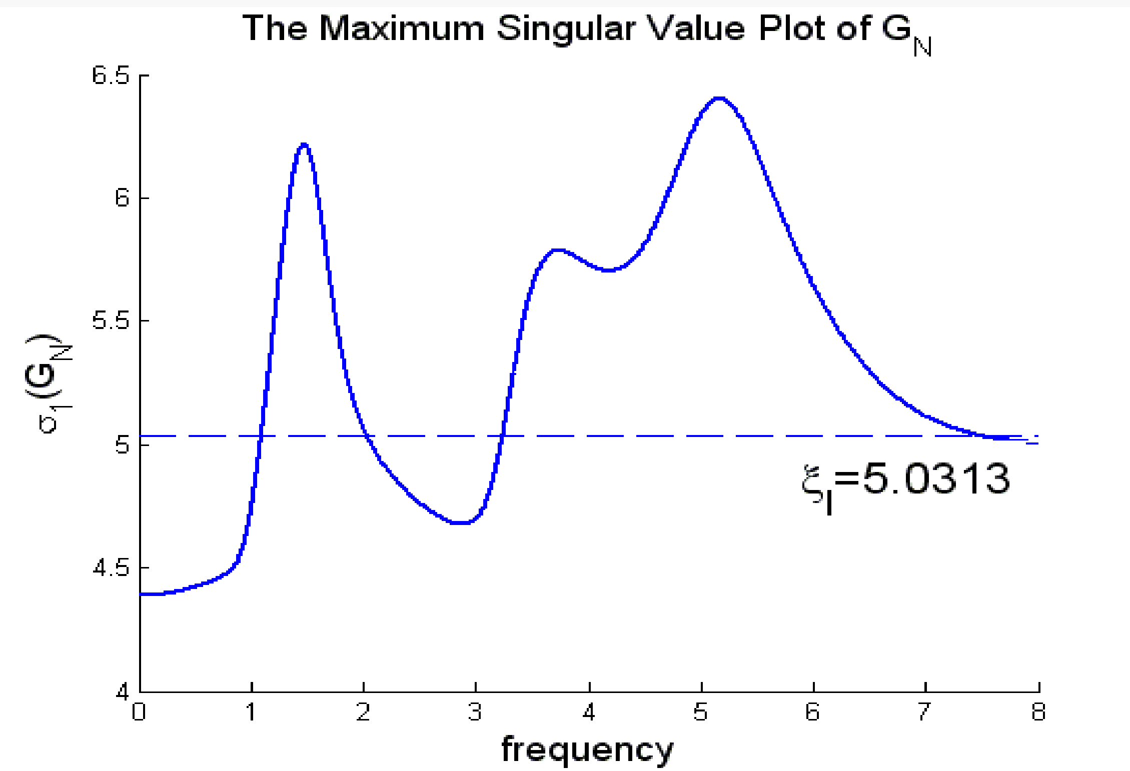
$$\begin{bmatrix} u \\ v \end{bmatrix} = x_0^{(i)},\ \ \omega=\omega^{(i)},\quad \xi=\xi_l.$$

denote the solution with $(\tilde{u}^{(i)},\tilde{v}^{(i)},\tilde{\omega}^{(i)},\tilde{\xi}^{(i)})$

- Set $\|G(j\omega)\|_{\mathcal{H}_\infty} := \max_{1\le i\le p}\tilde{\xi}^{(i)}.$

Second Algorithm for H$_\infty$ Norm Computation for TDS – Prediction Step





Second Algorithm for H_∞ Norm Computation for TDS – Prediction Step

$$\xi_l := \max\{\sigma_1(G(0)), \sigma_1(D), \text{tol}, \sigma_1(G_N(j\omega_t))\}$$

$$\xi := \xi_l(1 + 2\ \text{tol})$$

$$\|G_N\|_{\mathcal{H}_\infty} = \frac{\xi+\xi_l}{2}$$

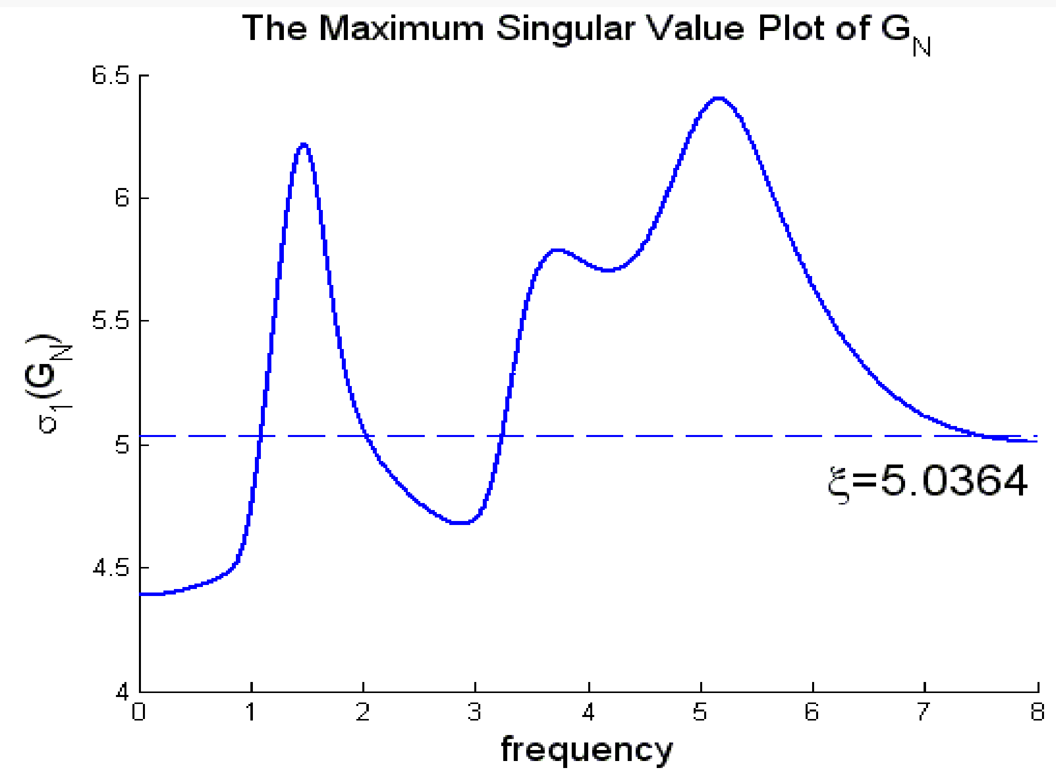
$$\{\lambda\} = \text{eig}(L_\xi^N)$$

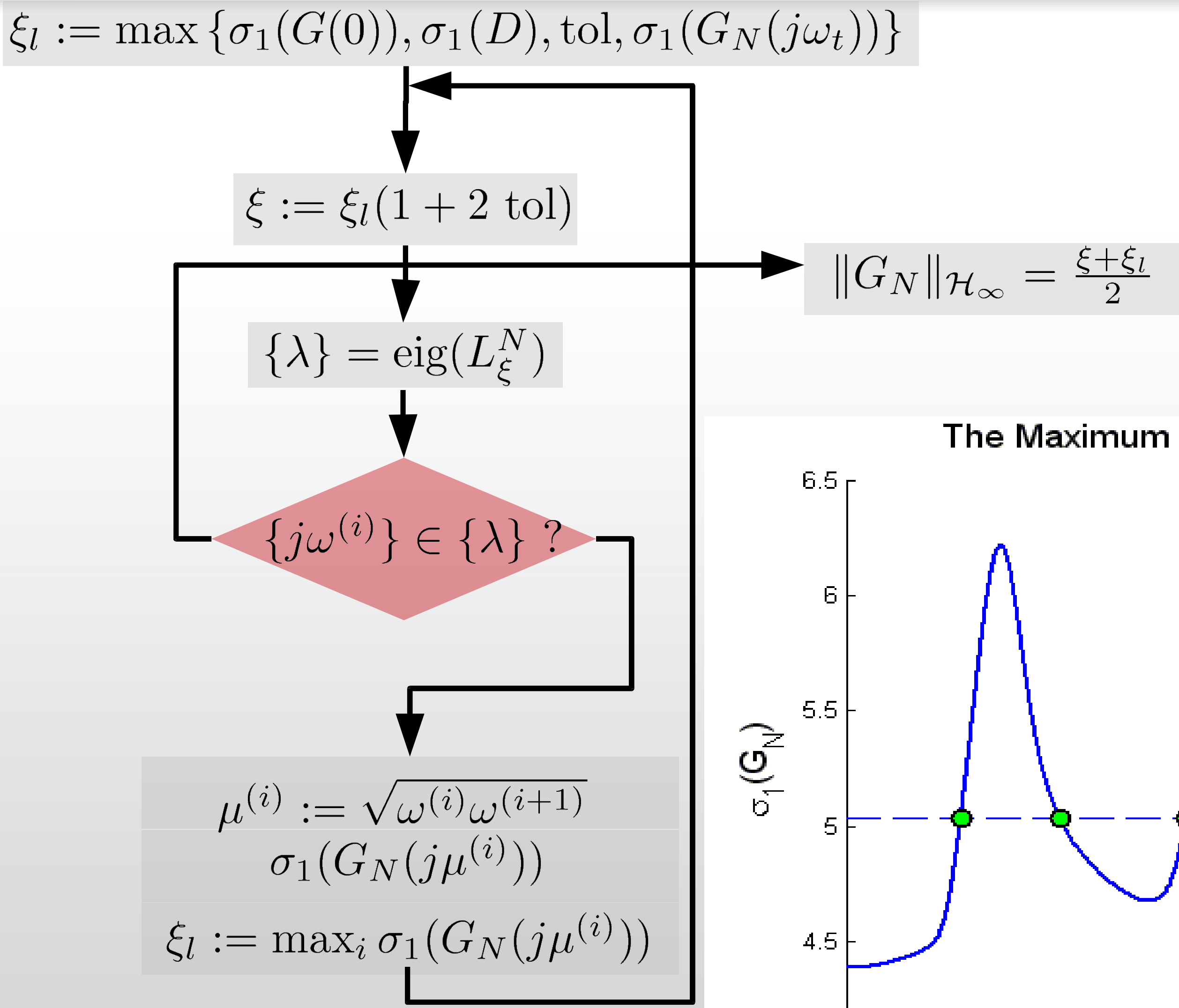
$$\{j\omega^{(i)}\} \in \{\lambda\}\ ?$$

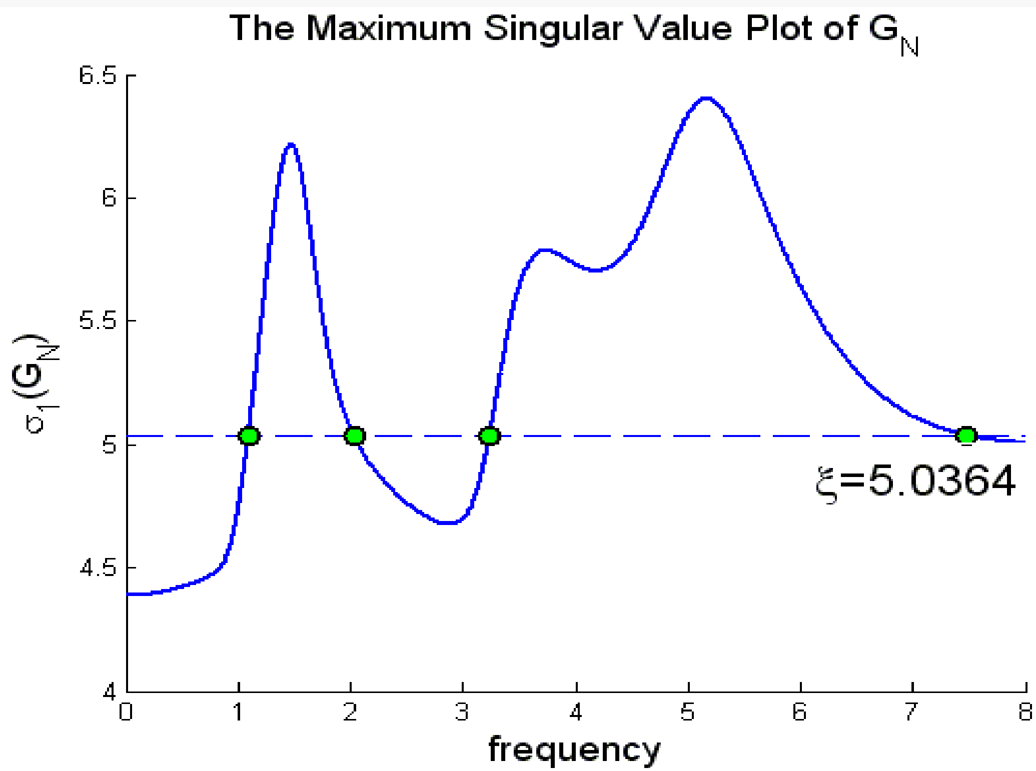
$$\mu^{(i)} := \sqrt{\omega^{(i)}\omega^{(i+1)}}$$

$$\sigma_1(G_N(j\mu^{(i)}))$$

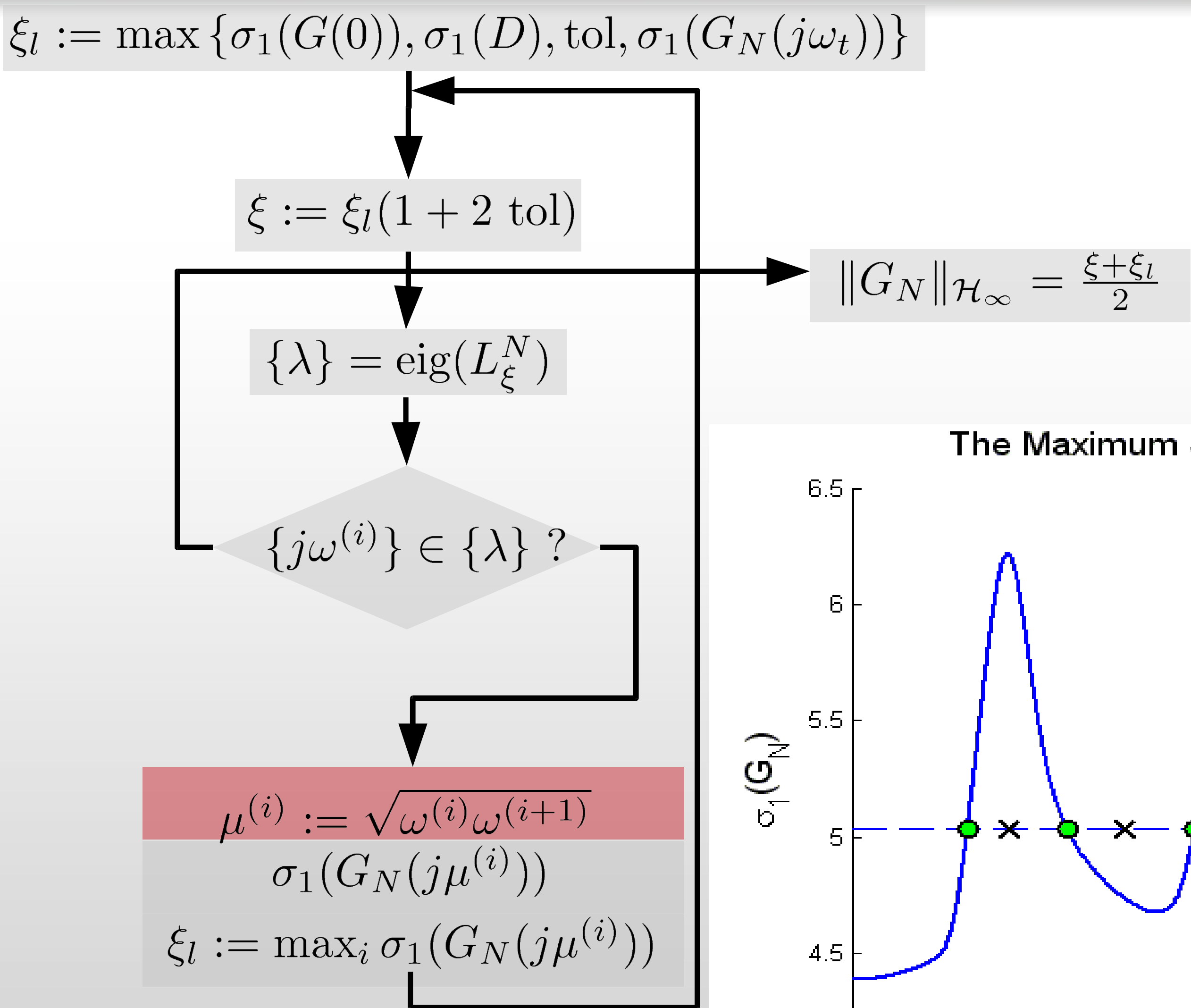
$$\xi_l := \max_i \sigma_1(G_N(j\mu^{(i)}))$$

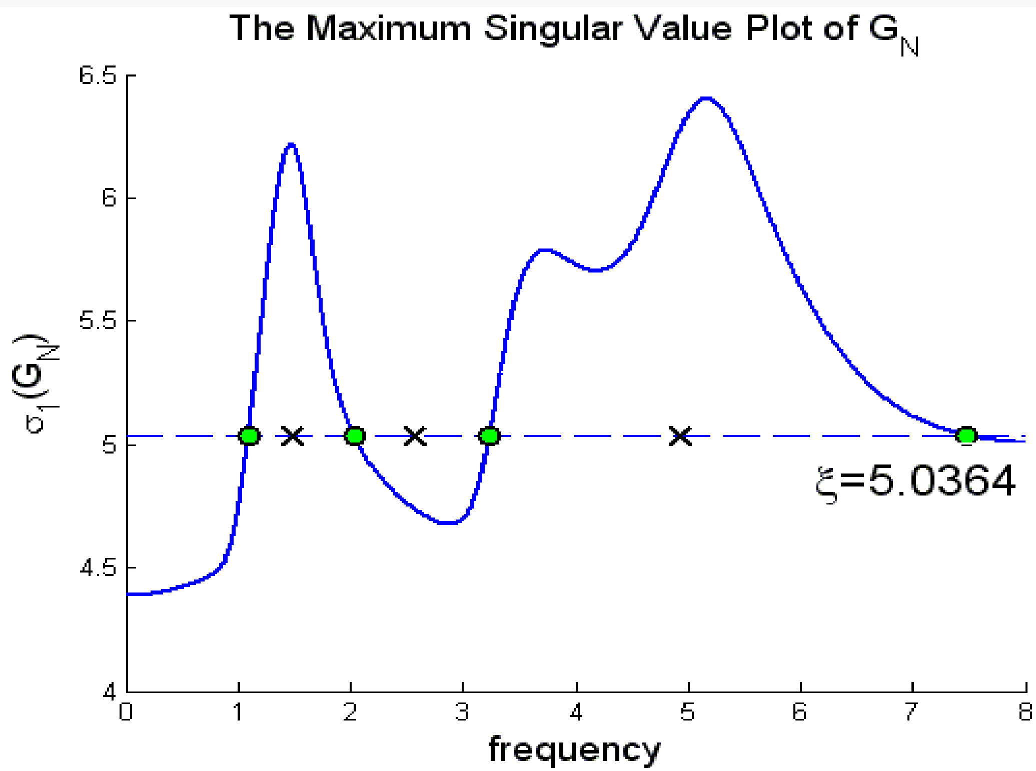



$\xi_l := \max\{\sigma_1(G(0)), \sigma_1(D), \text{tol}, \sigma_1(G_N(j\omega_t))\}$
$\xi := \xi_l(1 + 2\ \text{tol})$
$\|G_N\|_{\mathcal{H}_\infty} = \frac{\xi+\xi_l}{2}$
$\{\lambda\} = \text{eig}(L_\xi^N)$
$\{j\omega^{(i)}\} \in \{\lambda\}$?
$\mu^{(i)} := \sqrt{\omega^{(i)}\omega^{(i+1)}}$
$\sigma_1(G_N(j\mu^{(i)}))$
$\xi_l := \max_i \sigma_1(G_N(j\mu^{(i)}))$

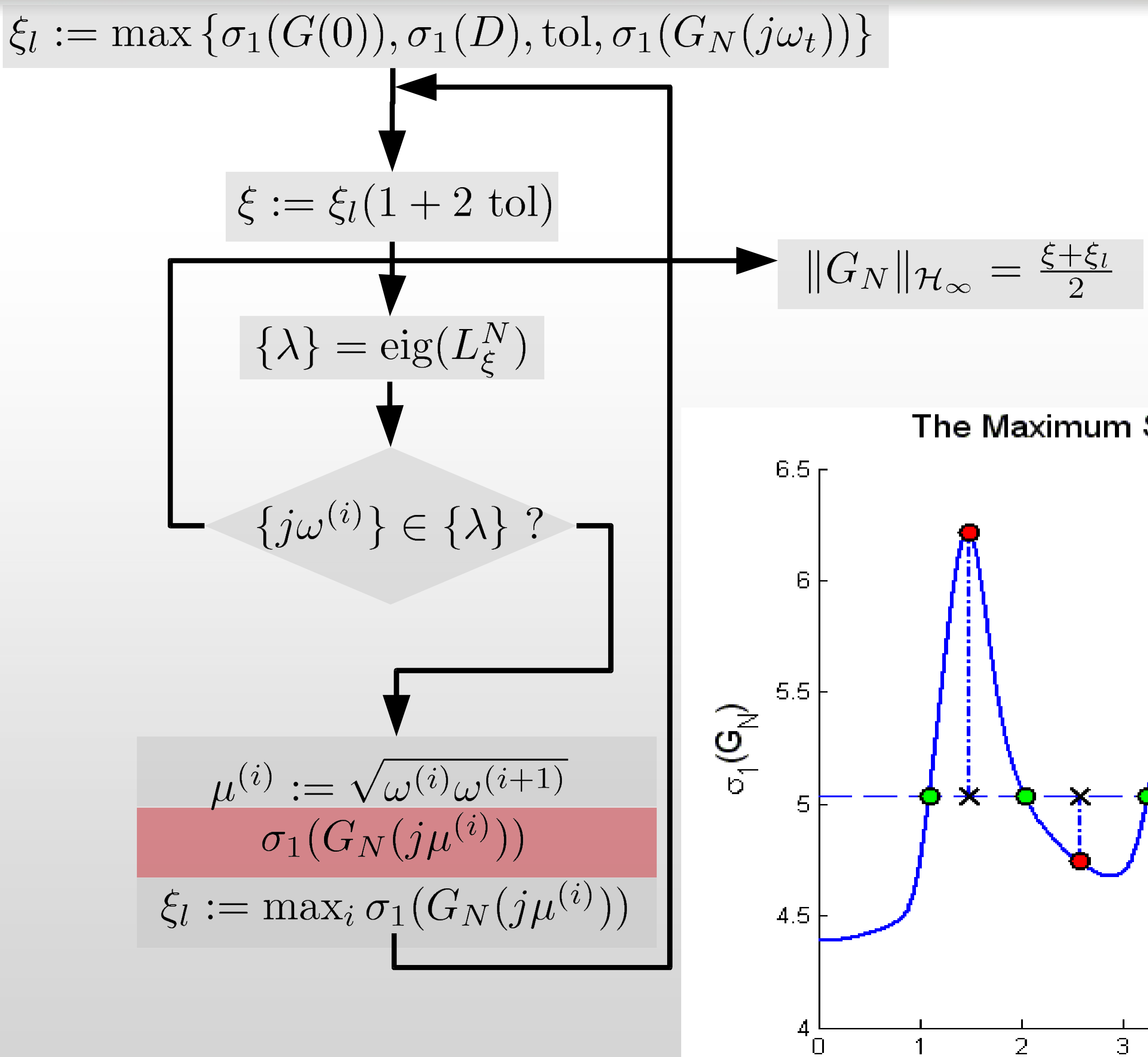

The Maximum Singular Value Plot of G_N
$\sigma_1(G_N)$
6.5
6
5.5
5
4.5
4
0
1
2
3
4
5
6
7
8
frequency
ξ=5.0364

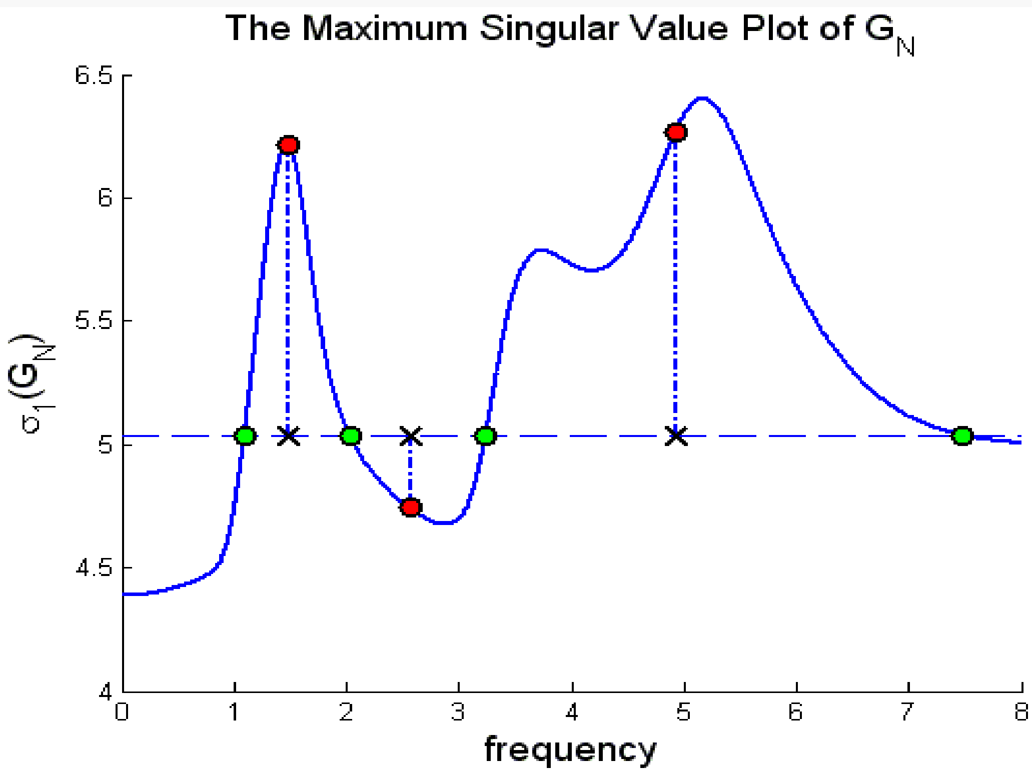
Second Algorithm for H_∞ Norm Computation for TDS – Prediction Step



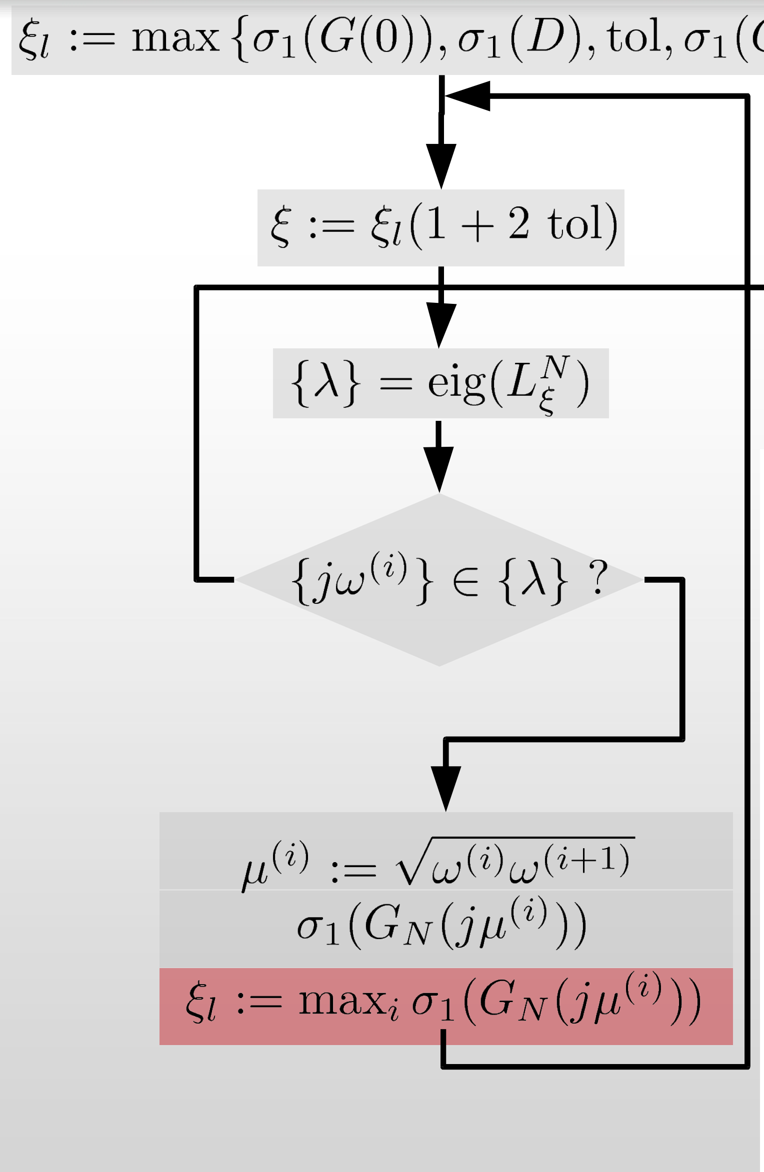


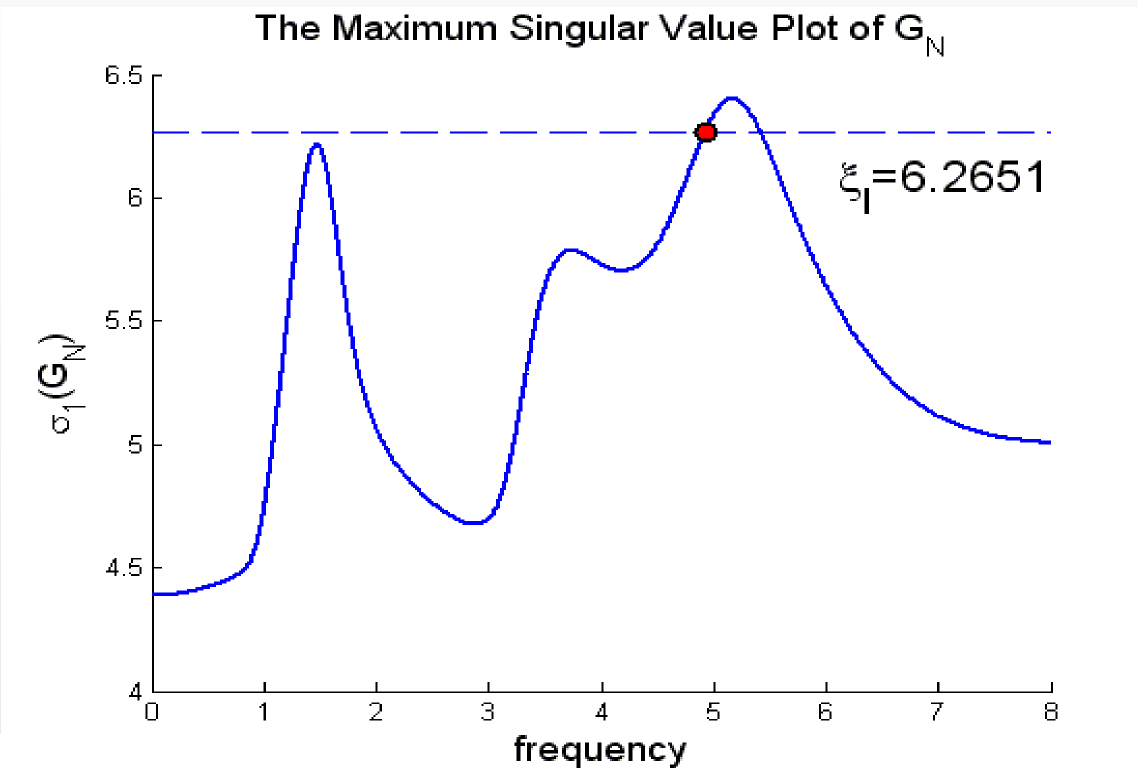
Second Algorithm for H_∞ Norm Computation for TDS – Prediction Step



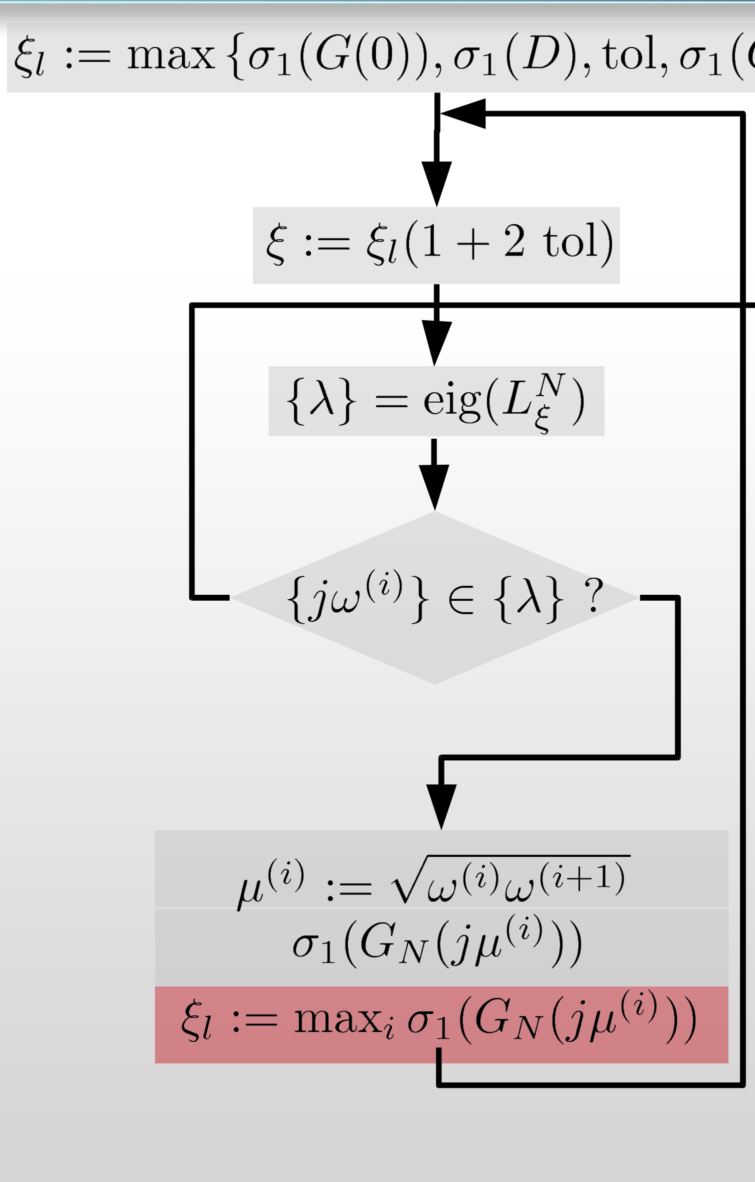


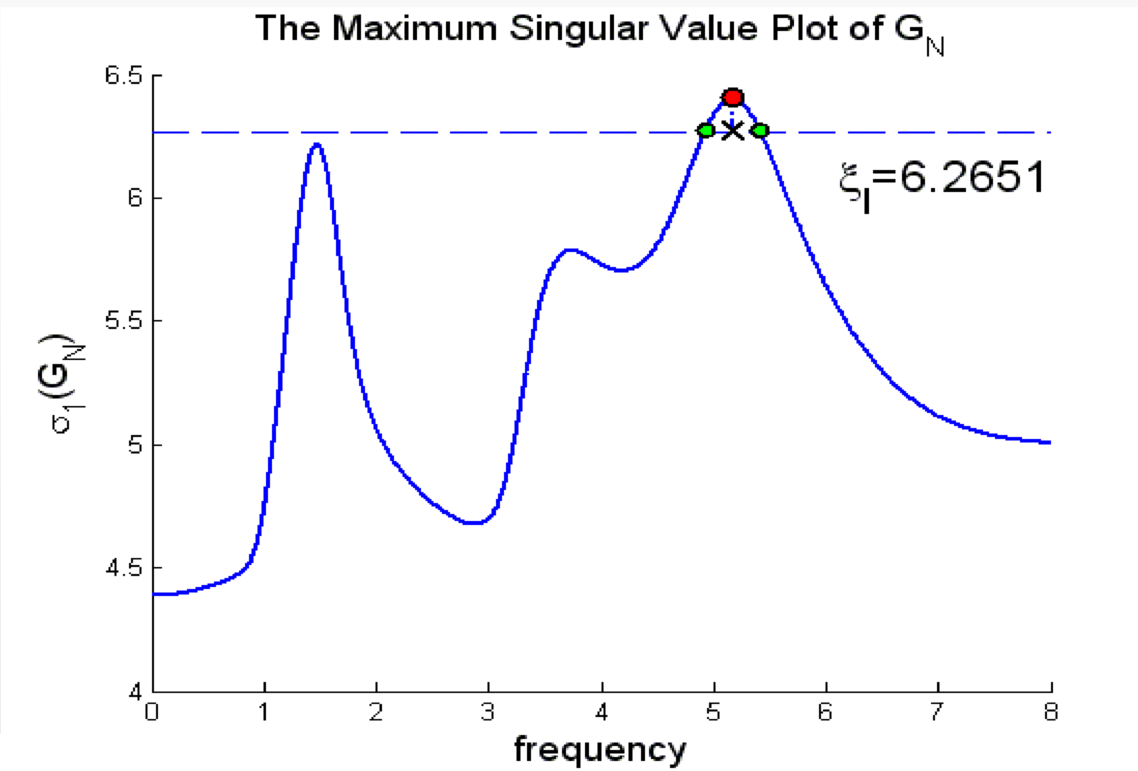
Second Algorithm for H_∞ Norm Computation for TDS – Prediction Step



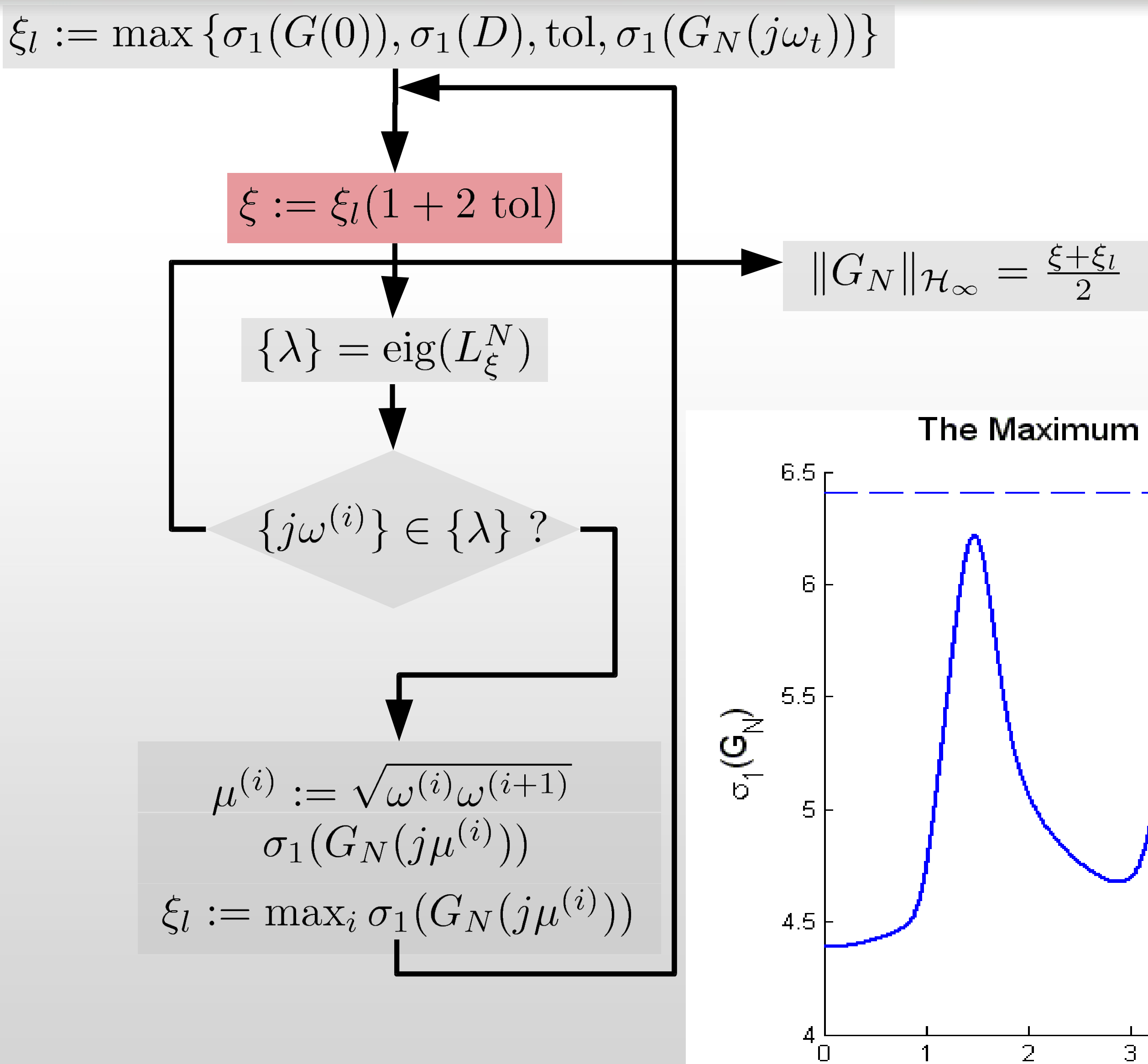


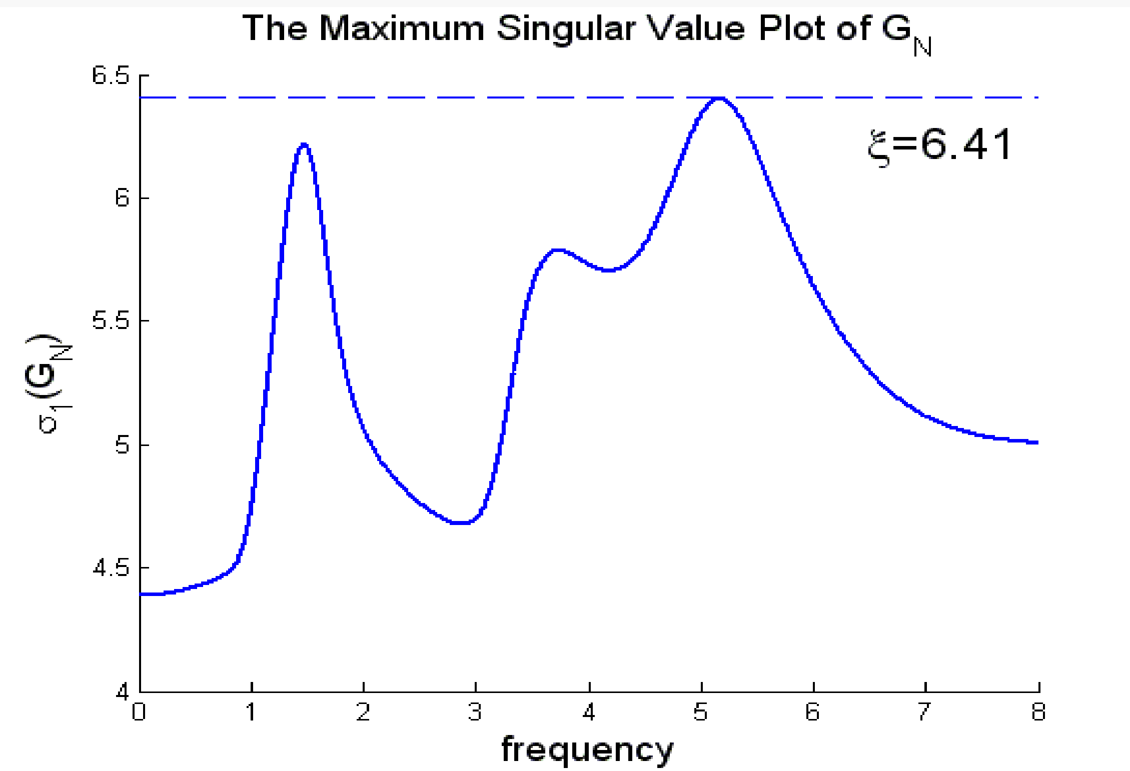
Second Algorithm for H_∞ Norm Computation for TDS – Prediction Step



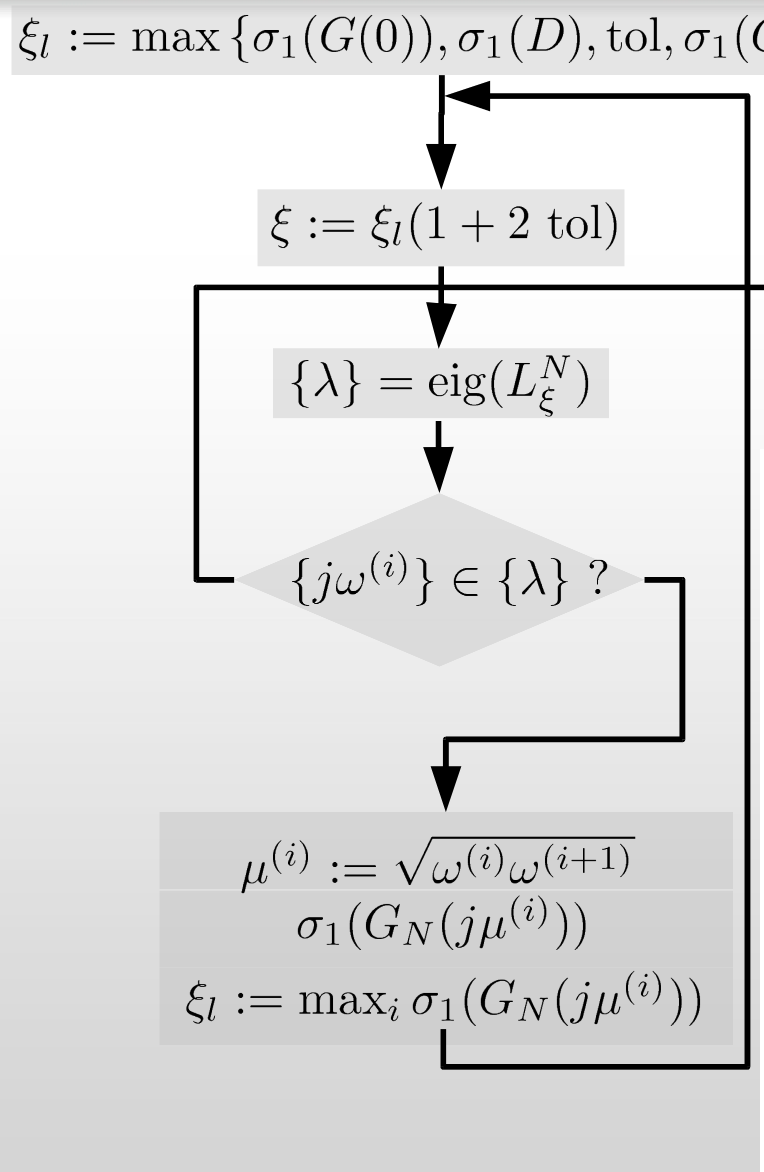


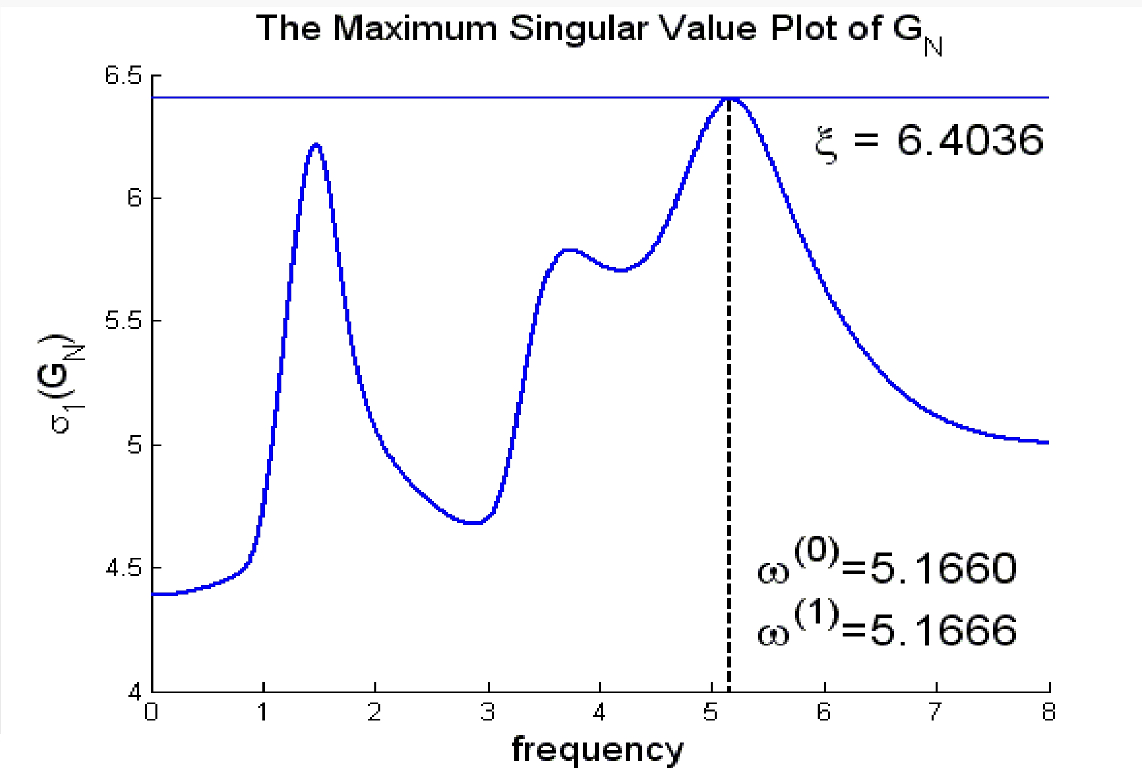
Second Algorithm for H_∞ Norm Computation for TDS – Prediction Step





Second Algorithm for H_∞ Norm Computation for TDS – Prediction Step





Second Algorithm for H_∞ Norm Computation for TDS – Correction Step

- The prediction step is quadratically convergent.

- The correction step is same as in the first algorithm.

Remarks

- Computation of G_N is needed only for specific frequencies and requires solving generalized eigenvalue problem with matrix size 2N+1
- The numerical method for computing H_∞ norm can be used for computing L_∞ norm of the time-delay system without any modification.

Remarks

- The definition of G_N interprets: $p_N(t,\lambda) \approx e^{\lambda\, t}$ for $t \in [-\tau_{\max}, \tau_{\max}]$

$$G_N(j\omega) = C\left(j\omega I - A_0 - \sum_{i=1}^{m} A_i p_N(-\tau_i;\ j\omega)\right)^{-1} B + D$$

- Note that the use of the well-known Pade approximation for the time- delay will cause numerically bad-scaled matrix in L_ζ^N due to the different magnitudes in the Pade coefficients.
- The Pade approximation depends on the time-delay and for multiple delays, each delay is approximated separately which will increase the L_ζ^N dimension considerably. However, the term $p_N(t,\lambda)$ approximates multiple delays with a single term.

Example

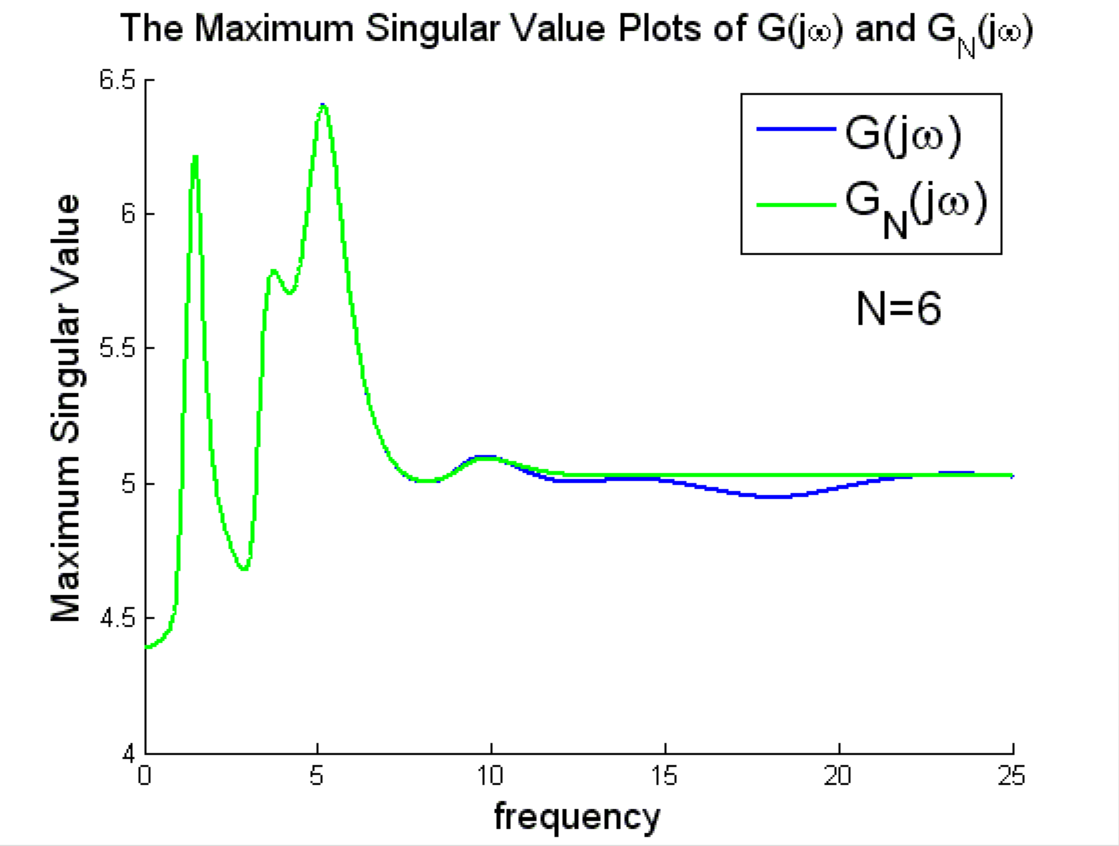
- The time-delay system G has dimensions:

$$G(s) = C^{4\times 10}\left(sI - A_0^{10\times 10} - \sum_{i=1}^{7} A_i^{10\times 10} e^{-\tau_i s}\right)^{-1} B^{10\times 2} + D^{4\times 2}$$

and delays

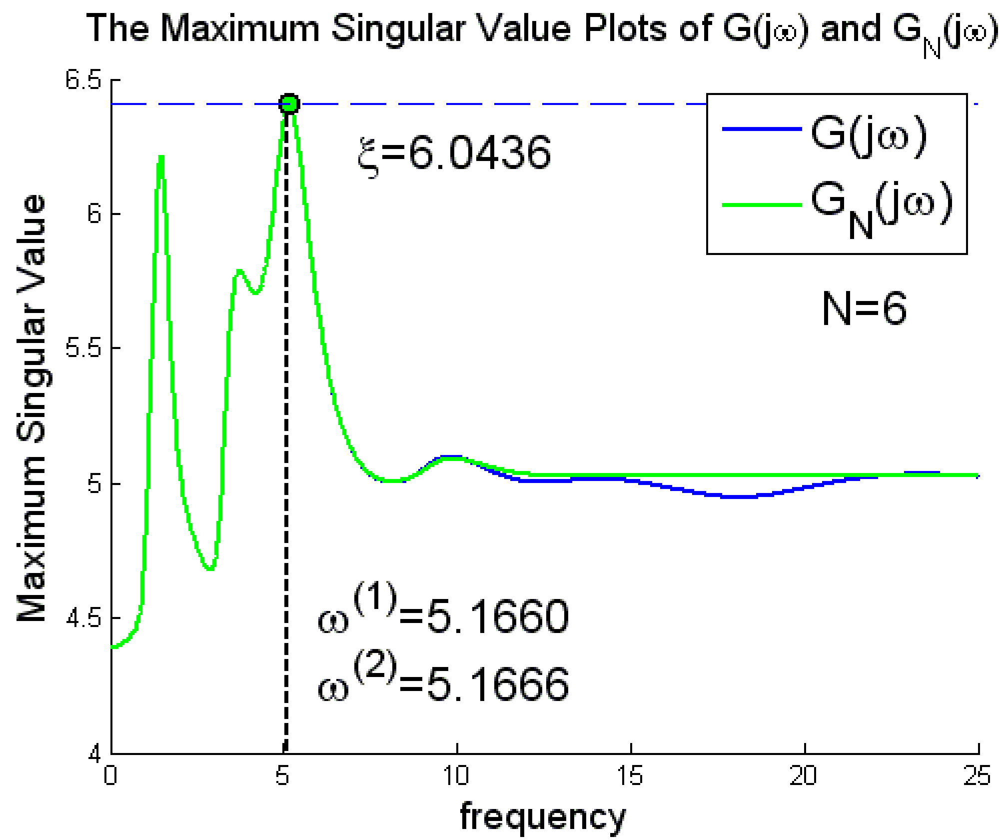
$$\tau_1 = 0.1,\ \tau_2 = 0.2,\ \tau_3 = 0.3,\ \tau_4 = 0.4,\ \tau_5 = 0.5,\ \tau_6 = 0.6,\ \tau_7 = 0.8.$$

Example

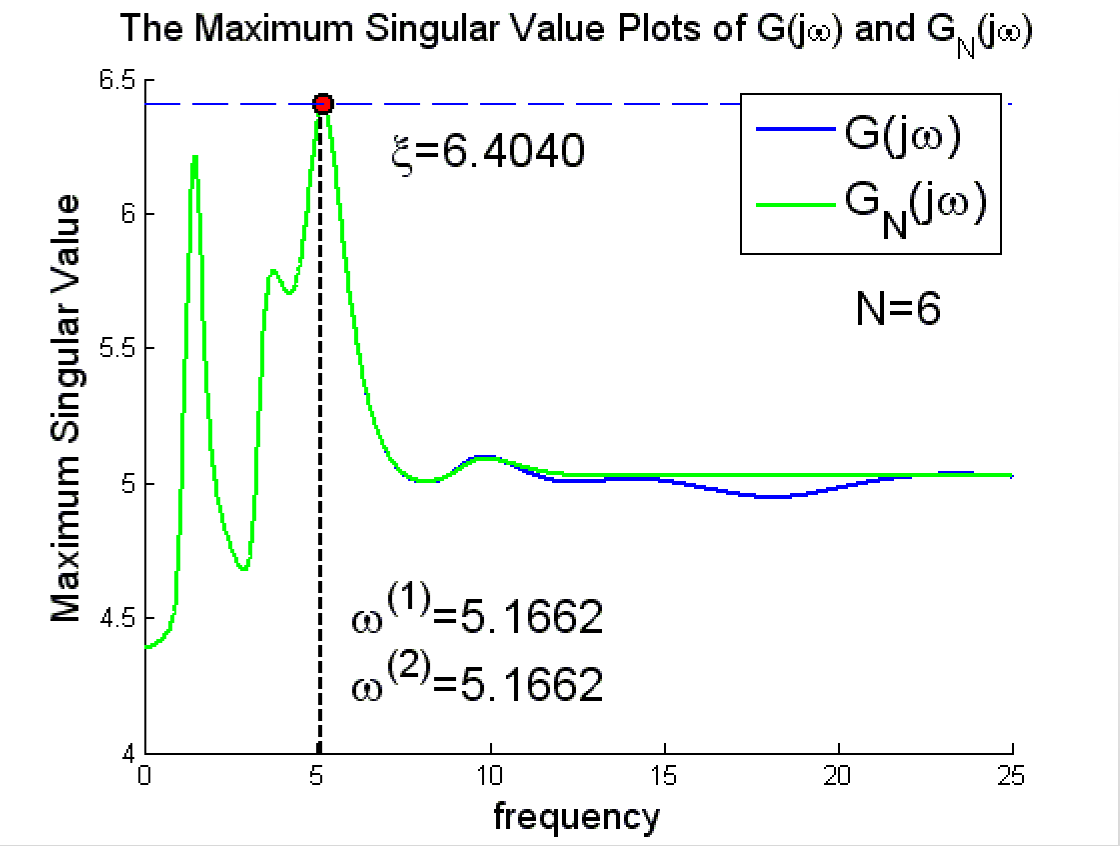


The Maximum Singular Value Plots of G(jω) and G_N(jω)
G(jω)
G_N(jω)
N=6
Maximum Singular Value
frequency
6.5
6
5.5
5
4.5
4
0
5
10
15
20
25

After Prediction Step



After Correction Step



Concluding Remarks

- The connection between the singular values of time-delay systems and the eigenvalues of infinite dimensional operator L_ζ is established
- A numerically stable method to compute H_∞ norm of time-delay system with arbitrary number of delays is given:
  - H_∞ norm prediction by discretization of the L_ζ^N
  - H_∞ norm correction using the equations based on nonlinear eigenvalue problem
- The algorithms are easily extendable to the systems with distributed delays