Fixed-Order H-infinity Optimization of Time-Delay Systems

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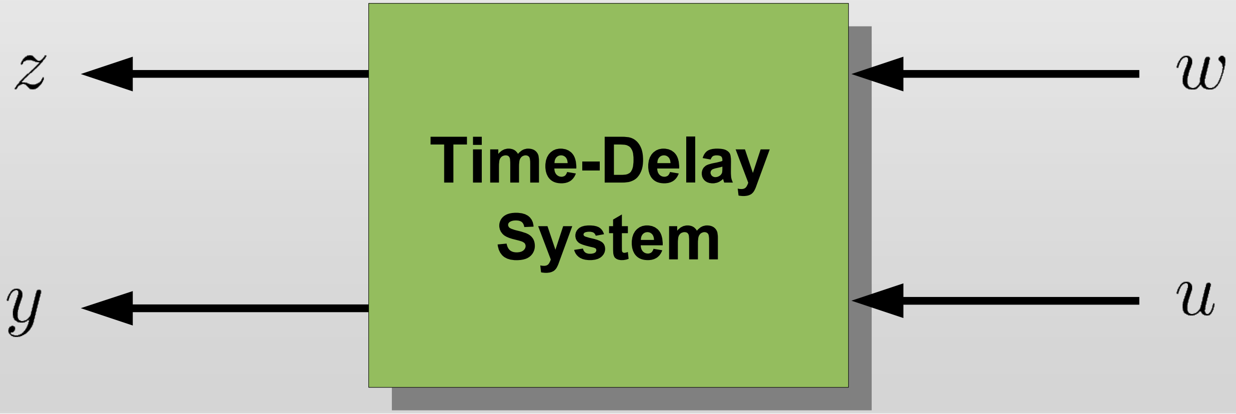
Plant Definition

The state-space equations of the generalized plant G are

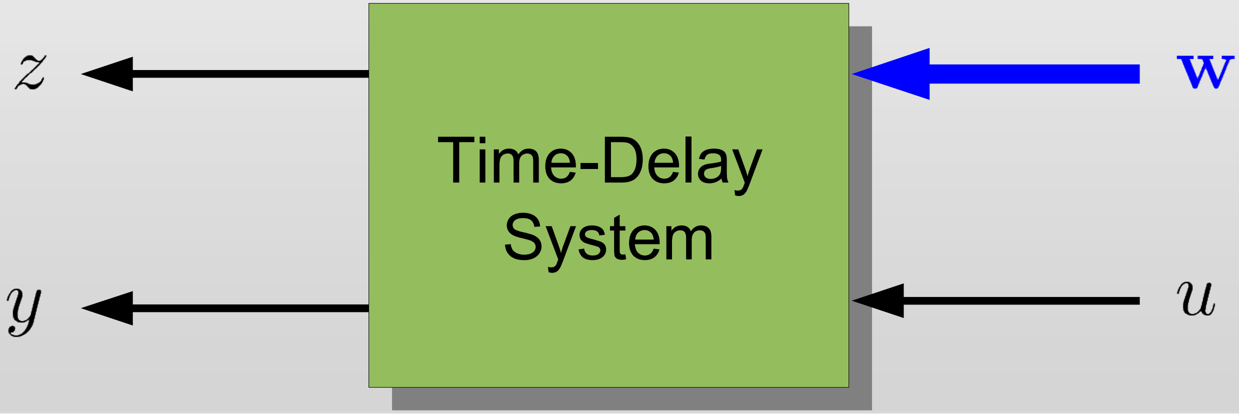
$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t-\tau_{m+1}) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t-\tau_{m+1}), A_0 \in \mathbb{R}^{n\times n}\end{aligned}$$



Plant Definition

Input functions are disturbances

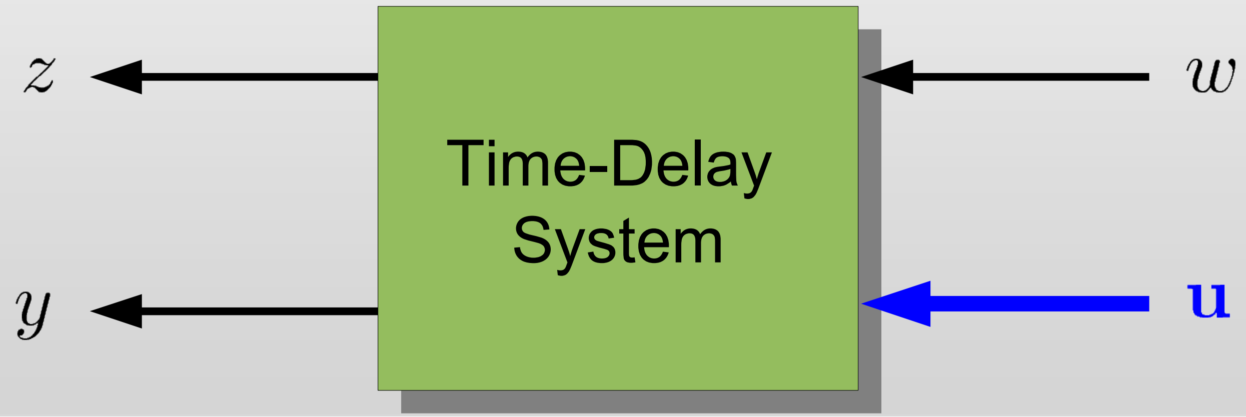
$$
\begin{aligned}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + B_1 \mathbf{w}(\mathbf{t}) + B_2 u(t-\tau_{m+1}) \\
z(t) &= C_1 x(t) + D_{11}\mathbf{w}(\mathbf{t}) + D_{12} u(t) \\
y(t) &= C_2 x(t) + D_{21}\mathbf{w}(\mathbf{t}) + D_{22} u(t-\tau_{m+1})
\end{aligned}
$$



Plant Definition

Input functions are control signals

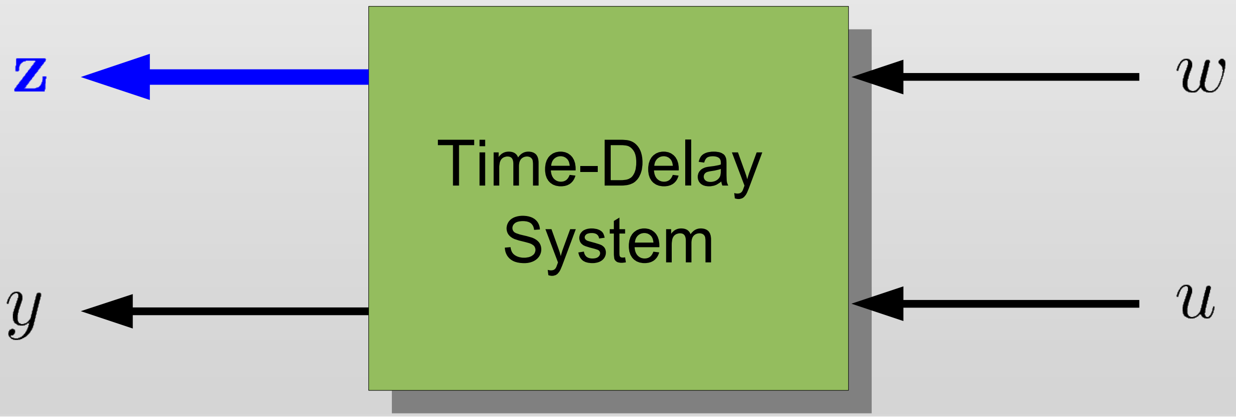
$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + B_1 w(t) + B_2 \mathbf{u}(\mathbf{t} - \tau_{\mathbf{m+1}}) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} \mathbf{u}(\mathbf{t}) \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} \mathbf{u}(\mathbf{t} - \tau_{\mathbf{m+1}}) \end{aligned}$$



Plant Definition

Output functions are controlled signals

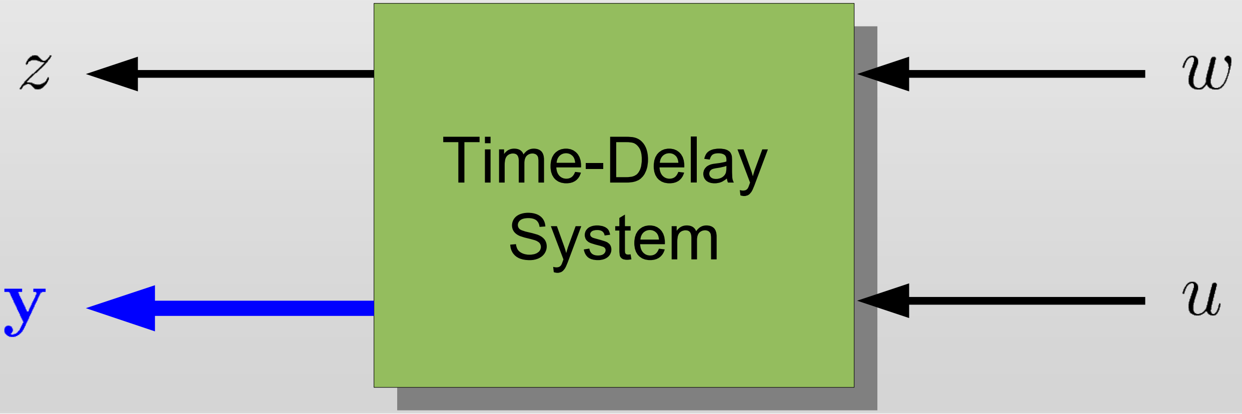
$$
\begin{aligned}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t-\tau_{m+1}) \\
\mathbf{z(t)} &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t-\tau_{m+1})
\end{aligned}
$$



Plant Definition

Output functions are measured signals

$$
\begin{aligned}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t-\tau_{m+1}) \\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
\mathbf{y}(\mathbf{t}) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t-\tau_{m+1})
\end{aligned}
$$



Controller structure

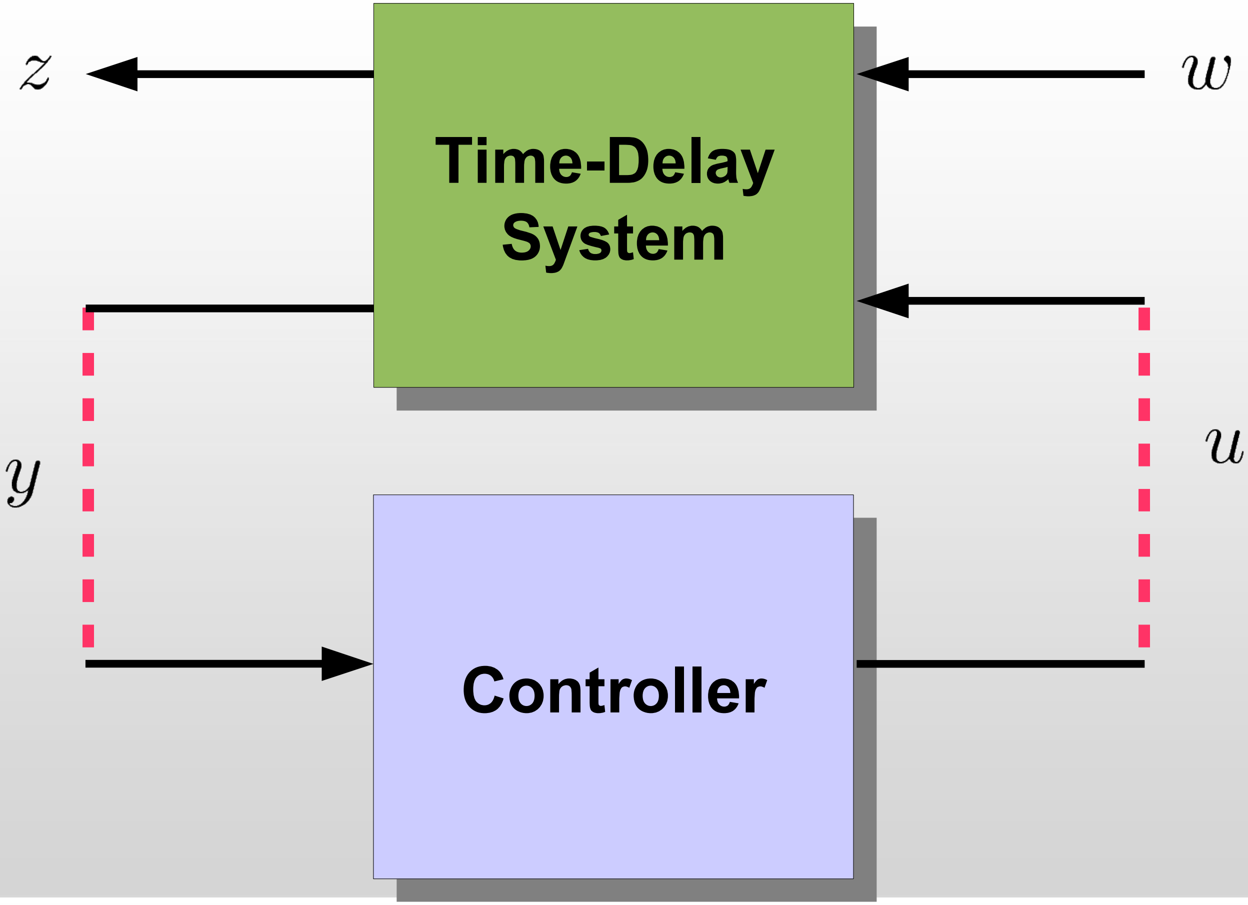
The controller has the following structure

$$\begin{aligned} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\ u(t) &= C_K x_K(t) \end{aligned}$$

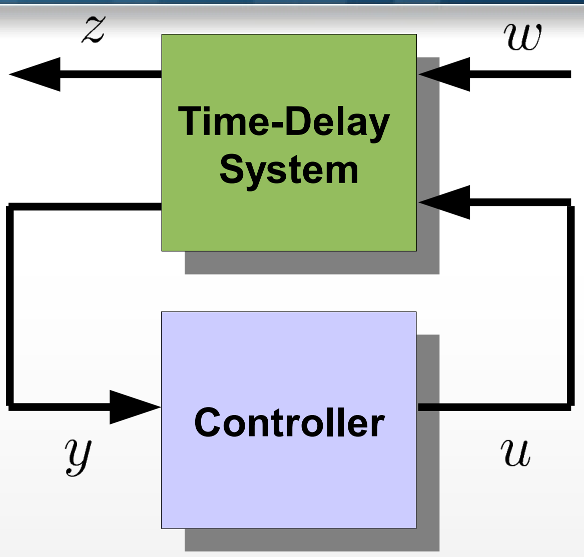
The controller order is n_K where $A_K \in \mathbb{R}^{n \times n}$. Note that the order of the controller is a design parameter.

Closing the loop

Connect the generalized plant and the controller



Closed-loop equations



We can write the closed-loop equations as

$$\dot{x}_{cl}(t) = A_{cl,0}x_{cl}(t) + \sum_{i=1}^{m+1} A_{cl,i}x_{cl}(t-\tau_i) + B_{cl}w(t)$$

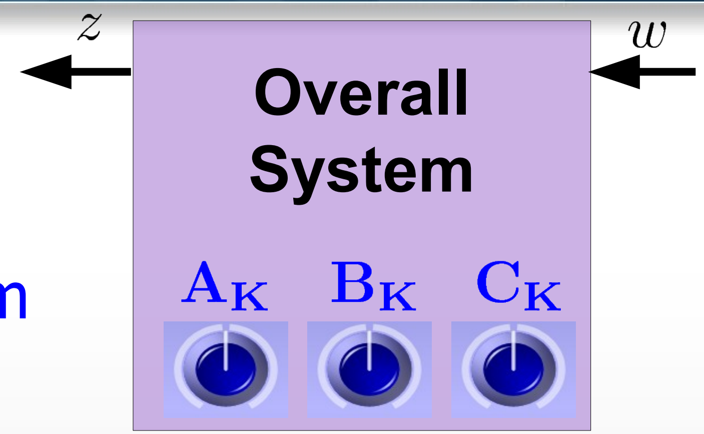
$$z(t) = C_{cl}x_{cl}(t) + D_{cl}w(t)$$

and the transfer function from w to z as

$$T_{zw}(s) = C_{cl}\left(sI - A_{cl,0} - \sum_{i=1}^{m+1} A_{cl,i}e^{-\tau_i s}\right)^{-1} B_{cl} + D_{cl}$$

The controller matrices can be tuned to achieve certain objectives.

Problem definition



We would like to minimize H-infinity norm from w to z

$$\|T_{zw}\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma\left(C_{cl}\left(j\omega I - A_{cl,0} - \sum_{i=1}^{m+1} A_{cl,i} e^{-j\tau_i \omega}\right)^{-1} B_{cl} + D_{cl}\right)$$

by a fixed-order controller

while keeping the overall system stable

$$\alpha\left(\sum_{i=0}^{m} A_{cl,i} e^{-\tau_i s}\right) < 0$$

α denotes the spectral abscissa, max real parts of eigenvalues

Why designing a fixed-order H-infinity controller?

- Fixed-order controllers are
  - cheap and easy to implement in hardware
  - non-restrictive in sampling rate and bandwidth
  - good alternative to controller order reduction and performance degradation check in design
- H-infinity controllers
  - stabilize the closed-loop under model uncertainties
  - achieve design performance for a predefined set of input signals
  - reject a set of disturbances (input disturbances, measurement noise)

Literature tells us

- For MIMO finite dimensional plants, H-infinity controller design
  - Optimal: solvable by standard methods when $n=n_K$ (Riccati-DGKF, LMI-Gahinet/Apkarian). This is restrictive for high-order plants.
  - Optimal, fixed-order: only for certain types of closed-loop functions (Nagamune)
  - Fixed-order: via non-smooth, non-convex optimization methods (HIFOO-Overton et.al., Apkarian/Noll)

- For time-delay plants, there is no known method for fixed-order H-infinity controller design

Objective function

We would like to find a local minimizer of $\|T_{zw}\|_\infty$ where

$$\|T_{zw}\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma\left(C_{cl}\left(j\omega I - A_{cl,0} - \sum_{i=1}^{m+1} A_{cl,i}e^{-j\tau_i\omega}\right)^{-1} B_{cl} + D_{cl}\right)$$

Note the objective function is

- not everywhere differentiable, may even be not Lipschitz continuous
- smooth almost everywhere

Optimization problem

Optimization method looks for a local minimizer of $\|T_{zw}\|_\infty$

where

$$\|T_{zw}\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma\left(C_{cl}\left(j\omega I - A_{cl,0} - \sum_{i=1}^{m+1} A_{cl,i} e^{-j\tau_i \omega}\right)^{-1} B_{cl} + D_{cl}\right)$$

Similarly to HIFOO, the method implements a hybrid algorithm for nonsmooth, nonconvex optimization, based on the following elements:

- A quasi-Newton algorithm (BFGS) provides a fast way to approximate a local minimizer
- A local bundle method attempts to verify local optimality for the best point found by BFGS, and if this does not succeed,
- Gradient sampling attempts to refine the approximation of the local minimizer, returning a rough optimality measure.

More information on optimization method

- J. V. Burke, A. S. Lewis and M. L. Overton, “A robust gradient sampling algorithm for nonsmooth nonconvex optimization,” *SIAM Journal on Optimization*, vol.15, pp. 751–779, 2005.
- J.V. Burke, D. Henrion, A.S. Lewis and M.L. Overton, “HIFOO - A MATLAB Package for Fixed-Order Controller Design and H-infinity Optimization,” Proc. of ROCOND 2006, Toulouse, July 2006.
- S. Gumussoy and M.L. Overton, “Fixed-Order H-infinity Controller Design via HIFOO, a Specialized Nonsmooth Optimization Package,” Proc. of ACC, Seattle, 2008.
- S. Gumussoy, D. Henrion, M. Millstone and M.L. Overton, “Multiobjective Robust Control with HIFOO 2.0,” IFAC Symposium on Robust Control Design, Haifa, 2009.
- http://www.cs.nyu.edu/overton/software/hifoo/

Computation of H-infinity norm

Optimization method requires the computation of $\|T_{zw}\|_\infty$ and gradients with respect to controller parameters

where

$$\|T_{zw}\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma\left(C_{cl}\left(j\omega I - A_{cl,0} - \sum_{i=1}^{m+1} A_{cl,i}e^{-j\tau_i\omega}\right)^{-1} B_{cl} + D_{cl}\right)$$

We implemented a predictor-corrector type method to evaluate H-infinity norm of T_{zw}

- Prediction step: calculate the approximate H-infinity norm and corresponding frequencies
- Correction step: correct the approximate results from the predicted step

Note that derivatives with respect to controller parameters can be computed from the derivative of an individual singular value

Predicton step

[Thm] Let ζ> 0 be such that the matrix $D_\xi := D_{cl}^T D_{cl} - \xi^2 I$ is non-singular. Singular values of T_{zw} and eigenvalues of the Hamiltonian-like operator L_ζ of T_{zw} have the relation:

$$\sigma_i(T_{zw}(j\omega_0)) = \xi \iff L_\xi u = j\omega_0 \, u$$

[Corollary]

$$\|T_{zw}\|_\infty = \sup\{\xi \in \mathbb{R}_+ : \text{ operator } \mathcal{L}_\xi \text{ has an eigenvalue on the imaginary axis}\}$$

Predicton step: Properties of L_ζ

- infinite dimensional linear operator
- has infinitely many eigenvalues, finite on imaginary axis
- its spectrum symmetric wrt imaginary axis
- eigenvalues of L_ζ can be computed approximately by computing eigenvalues of the matrix L_ζ^N obtained by spectral methods (Breda et.al.)

$$\|T_{zw}\|_\infty \approx \sup\{\xi \in \mathbb{R}_+ : \text{ matrix } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$$

based on

$$\sigma_i(T_{zw}(j\omega_0) = \xi \;\leftrightsquigarrow\; \det(j\hat{\omega}_0 I - L_\xi^N) = 0$$

Correction step

We want to correct the approximate results from prediction step.

[Thm] Let ζ> 0 be such that the matrix $D_\xi := D_{cl}^T D_{cl} - \xi^2 I$

is non-singular. λ is an eigenvalue of L_ζ if and only if

$$h_\xi(\lambda) := \det H_\xi(\lambda) = 0$$

where

$$H_\xi(\lambda) := \lambda I - M_0 - \sum_{i=1}^{m+1} \left(M_i e^{-\lambda \tau_i} + M_{-i} e^{\lambda \tau_i} \right)$$

$$\sigma_i(T_{zw}(j\omega_0)) = \xi \iff L_\xi u = j\omega_0 u \iff h_\xi(j\omega_0) = 0$$

Correction step

Using the properties of $h_\xi(j\omega)$, $h_\xi(j\omega)=0, \quad h'_\xi(j\omega)=0$

$$\begin{cases} H(j\omega,\ \xi)\begin{bmatrix} u, \\ v \end{bmatrix} = 0, \quad n(u,v)=0 \\ \Im\left\{v^*\left(I+\sum_{i=1}^{m+1} A_{cl,i}\tau_i e^{-j\omega\tau_i}\right)u\right\}=0 \end{cases}$$

- Overdetermined system (4n+3 equations, 4n+2 unknowns)
- Exactly solvable, overdetermined due to symmetry of the spectrum
- Solved using Gauss-Newton algorithm (quadratically converging because residual in the solution is zero)

Computation of H-infinity Norm

Prediction step

For fixed N, determine

$$\sup\{\xi \in \mathbb{R}_+ : \text{ matrix } \mathcal{L}_\xi^N \text{ has an eigenvalue on the imaginary axis}\}$$

and determine the corresponding eigenvalues on the imaginary axis

(solved by a criss-cross search in two directions)

Correction step

Correct the results by solving the equations

$$\left\{ \begin{array}{l} H(j\omega,\ \xi) \left[\begin{array}{c} u, \\ v \end{array} \right] = 0, \quad n(u,v) = 0 \\ \Im \left\{ v^* \left(I + \sum_{i=1}^{m+1} A_{cl,i} \tau_i e^{-j\omega\tau_i} \right) u \right\} = 0 \end{array} \right.$$

Further details: S. Gumussoy and W. Michiels, "Computing H-infinity Norms of Time-Delay Systems," accepted to *CDC*, 2009.

Discussion on computational time

- The main step is the computing eigenvalues of the Hamiltonian matrix L_ζ^N (N: num of discretization points).
- The computational time proportional to

ODE ~ $(2n_1)^3$ (n_1: dim of ODE)

DDE ~ $(2n_2 \times 2N)^3$ (n_2: dim of DDE)

Benchmarking – Example 1

$$\begin{aligned} \dot{x}(t) &= -x(t) - 0.5x(t-1) + w(t) + u(t) \\ z(t) &= x(t) + u(t) \\ y(t) &= x(t) + w(t) \end{aligned}$$

- The time-delay system is stable and H-infinity norm is 0.72
- When n_K=1, our method finds the controller

$$\begin{aligned} \dot{x}_K(t) &= 3.61x_K(t) + 1.39y(t) \\ u(t) &= -0.83x_K(t) \end{aligned}$$

  achieving the closed-loop H-infinity norm 0.064
- For n_K=2, the norm is 0.021. For n_K=3, 0.020.

Benchmarking – Example 2

$$\dot{x}(t) = \begin{pmatrix} -4.4656 & -0.4271 & 0.4427 & -0.1854 \\ -0.8601 & -5.6257 & 0.8577 & -0.5210 \\ 0.9001 & -0.7177 & -6.5358 & 0.0417 \\ -0.6836 & 0.0242 & 0.4997 & -3.5618 \end{pmatrix} x(t) + \begin{pmatrix} 0.6848 & -0.0618 & 0.5399 & 0.5057 \\ 0.3259 & -0.3810 & 0.6592 & -0.0066 \\ 0.6325 & 0.3752 & 0.4122 & 0.7303 \\ 0.5878 & 0.9737 & 0.1907 & -0.8639 \end{pmatrix} x(t-3.2)$$

$$+ \begin{pmatrix} 0.9371 & -0.7859 & 0.1332 & 0.7429 \\ -0.8025 & 0.4483 & 0.6226 & 0.0152 \\ 0.0940 & 0.2274 & 0.1536 & 0.5776 \\ -0.1941 & 0.5659 & 0.8881 & -0.0539 \end{pmatrix} x(t-3.4) + \begin{pmatrix} 0.6576 & -0.8543 & -0.3460 & 0.6415 \\ -0.3550 & 0.5024 & 0.6081 & 0.9038 \\ 0.9523 & 0.6624 & 0.0765 & -0.8475 \\ -0.4436 & 0.8447 & -0.0734 & 0.4173 \end{pmatrix} x(t-3.9)$$

$$+ \begin{pmatrix} 1 & 0 \\ -1.6 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} w(t) + \begin{pmatrix} 0.2 \\ -1 \\ 0.1 \\ -0.4 \end{pmatrix} u(t-0.2)$$

$$z(t) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0.1 & 1 \\ -1 & 0.2 \end{pmatrix} w(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} -2 & 0.1 \end{pmatrix} w(t) + 0.4u(t-0.2)$$

Benchmarking – Example 2

- The time-delay system is stable and H-infinity norm is 1.3907
- When n_K=1, our method finds the controller

$$
\begin{aligned}
\dot{x}_K(t) &= -0.712x_K(t) - 0.1639y(t) \\
u(t) &= -0.2858x_K(t)
\end{aligned}
$$

achieving the closed-loop H-infinity norm 1.2606.

- For n_K=2, the norm is 1.2573. For n_K=3, 1.2505.

Concluding Remarks

- Fixed-order H-infinity controller design via non-smooth, non-convex optimization method is successfully implemented for a class of time-delay systems
- Our method allows to choose the controller order as desired
- This method is a promising alternative to controller order reduction methods
- Extension to general time-delay systems and allowing the structure in the controller are future research directions.
- Optimization based approach is essentially an approach for parameter tuning, thus applicable to a very broad class of problems, e.g. also controllers featuring delays $\sum_{k_i} y(t-\tau_i)$